Mathematics on a TI-84/CE
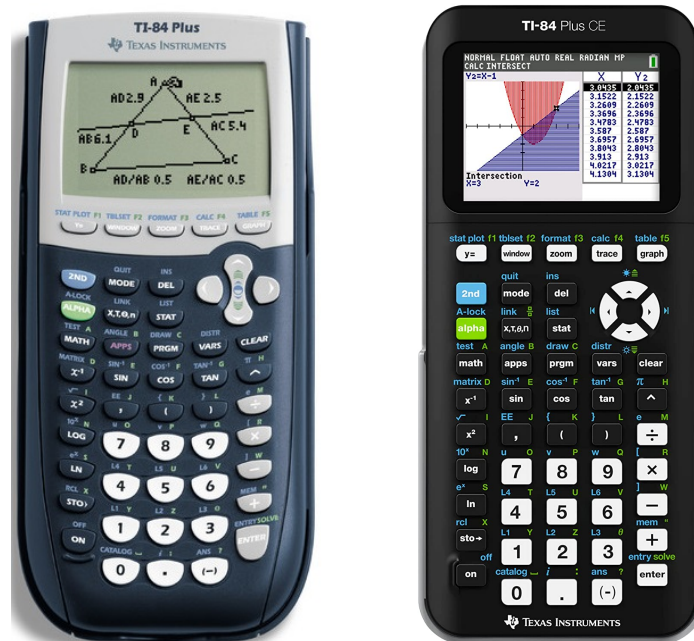
Volume 2: Calculus

Chapters $9-16$

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Contents

Mathematics on a TI-84/CE

Volume 1 of this book contains the basic topics: Graphics Calculators and Mathematics; Getting Started; Coordinate Geometry; Inequalities and Linear Programming; Fitting Curves to Data 1 – Calculator Functions; Population Modelling 1 – Exponential Growth; Financial Mathematics 1 – Compound Interest; and Probability and Statistics 1 – Descriptive Statistics. The Volume 1 Supplement contains extra activities for Coordinate Geometry and Probability and Statistics 1.

Volume 3 of this book contains the advanced topics: Sequences and Series; Probability and Statistics 2 – Probability Distributions and Hypothesis Testing; Matrices and Vectors; Population Modelling 3 – Matrix Models; Fitting Curves to Data 2; Financial Mathematics 2 – TVM Calculations; Complex Numbers; and Programming.

Volume 4 of this book contains 28 Mathematics labs or projects, together with a Lab Manual for teachers/instructors.

Calculator versions

Currently (early 2022), TI-84 calculators come in two versions: the TI-84Plus and the more recent TI-84CE. The main difference is that the CE screen has much higher resolution. It also has colour but I have done most of the screens in black and white to avoid the need for colour printers or photocopiers. Calculations, screenshots and figures were done on a TI-84CE in CLASSIC mode.

Some programs have had to be changed for the CE because of the different screen: I usually append 'CE' to the program name to indicate this.

All the programs here are available at *www.XXX*.

9 Functions and their Graphs

9.1 Introduction

Functions are integral to Mathematics, and the building blocks for mathematical modelling. The use of graphics calculators changes the way we treat functions. No more slaving away for 15 or 20 minutes to plot the graph of even a relatively simple function every time we meet a different transformation of it. Now most functions can be graphed quickly, so that time can be spent looking at and understanding the different transformations and asking how particular functions might be used in modelling.[1] Questions change from *Graph the function* $y = \sinh\big(\ln\big(x^2 + \sin(x)\big)\big)$ to *What does this function tell you about this particular system?*[2]

What follows here are some aspects of functions in general, with some graphics, then a graphical exploration of a number of useful and interesting functions with exercises and solutions. While the emphasis is on graphing functions using the basic graphics commands on the calculator, *Appendix 1: Spinners* shows how the use of calculator programs can make function exploration even easier and fun.[3] One or two other useful programs are also introduced.

Another aspect demonstrated here is how easy it is to incorporate graphics screens into teaching material, hardcopy or online. Details are given in Appendix 2.

9.1.1 Setting up for graphing

Press $\boxed{\text{mode}}$ and set up your calculator as shown below left and centre for a TI-84Plus, below right for a TI-84CE. You can use either MATHPRINT or CLASSIC.

9.1.2 Australian Curriculum

References are given here to the corresponding topics in the Australian Curriculum. Specific references are given to the texts *Nelson Senior Maths Methods 11* (NSM11) and *Nelson Senior Maths Methods 12* (NSM12) used in the ACT.

The material here in Sections 2–5 is directly relevant to the topic *Functions and graphs*, Chapter 4 in NSM11.

Section 9.3.4 here on periodic functions is relevent to the topic *Trigonometric functions and graphs*, Chapter 6 in NSM11. Several spinners (Appendix 1) are also relevant.

Section 9.3.1 here on exponential and logarithmic functions is relevent to part of the topic *Derivatives, exponential and trigonometric functions*, Chapter 1 in NSM12, and to the topic *Logarithmic functions*, Chapter 7 in NSM12.

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[1] I'm an applied mathematician.

[2] The function looks just like a wide 2D funnel!

[3] All the programs used in this topic are available at *www.XXX*.

9.2 Functions

Many common relationships involve two variables in such a way that the value of one of the variables depends on the value of the other. Consider the relationship between the area of a circle A and its radius r. This can be expressed by the equation $A = \pi r^2$: the value of A depends on the value of r. We call A the **dependent variable** and r the **independent variable**.

Of particular interest are relationships in which, to every value of the independent variable, there corresponds *one and only one* value of the dependent variable. We call this type of correspondence a **function**.

9.2.1 Definitions

A **function** f from a set X into a set Y is a correspondence that assigns to each element x in X exactly one element y in Y. We call y the **image** of x under f, and denote it by $f(x)$. The **domain** of f is the set X, and the **range** of f consists of all images of elements in X.

If, to each value y in its range, there corresponds exactly one value x in its domain, the function is said to be **one-to-one**. Moreover, if the range of f consists of all of Y, the function is called **onto**.
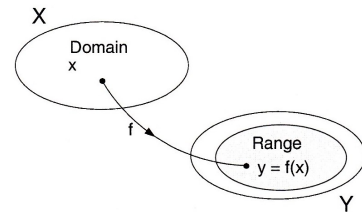
Remark: To begin with we work only with functions whose domains and ranges are sets of real numbers. We call such functions **real-valued functions** of a **real variable**.

Functions can be specified in a variety of ways. One way is an equation involving the dependent and independent variables. To evaluate a function described by an equation, we generally isolate the dependent variable on the left-hand side of the equation.

For example, the equation $x + 2y = 1$, written as

$$y = \frac{1-x}{2},$$

describes y as a function of x, and we can write this function as

$$f(x) = \frac{1-x}{2}.$$

This functional notation has the advantage of clearly identifying the dependent variable as $f(x)$ while at the same time providing a name 'f' for the function. The symbol $f(x)$ is read as 'f of x'. To denote the value of the dependent variable when $x = a$, we use the symbol $f(a)$. For example, the value of f when $x = 3$ is

$$f(3) = \frac{1-3}{2} = \frac{-2}{2} = -1.$$

In an equation that defines a function, the role of the variable x is simply that of a placeholder. For example, the function

$$f(x) = 2x^2 - 4x + 1$$

can be described by the form

$$f(\ ) = 2(\ )^2 - 4(\ ) + 1,$$

where parentheses are used instead of x. Therefore, to evaluate $f(-2)$, we simply place -2 in each set of parentheses:

$$f(-2) = 2(-2)^2 - 4(-2) + 1 = 2(4) + 8 + 1 = 17.$$

Remark: Although we generally use f as a convenient function name and x as the independent variable, we can use any symbols. For example, the following equations all define the same function.

$$
\begin{aligned}
f(x) &= x^2 - 4x + 7 \\
f(t) &= t^2 - 4t + 7 \\
g(s) &= s^2 - 4s + 7.
\end{aligned}
$$

Example 1   *Evaluating a function*

For the function $f(x) = x^2 - 4x + 7$, evaluate

(a) $f(3a)$.

We begin by writing the equation for f in the form

$$f(\ ) = (\ )^2 - 4(\ ) + 7.$$

Then, $f(3a) = (3a)^2 - 4(3a) + 7 = 9a^2 - 12a + 7$.

(b) $f(b-1)$

$$f(b-1) = (b-1)^2 - 4(b-1) + 7 = b^2 - 2b + 1 - 4b + 4 + 7 = b^2 - 6b + 12.$$

(c) $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$.

This one takes a bit longer.

$$
\begin{aligned}
\frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{\left((x+\Delta x)^2 - 4(x+\Delta x) + 7\right) - \left(x^2 - 4x + 7\right)}{\Delta x} \\
&= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 4x - 4\Delta x + 7 - x^2 + 4x - 7}{\Delta x} \\
&= \frac{2x\Delta x + (\Delta x)^2 - 4\Delta x}{\Delta x} \\
&= 2x - 4 + \Delta x.
\end{aligned}
$$

Exercises 1(a)–(f); 3(a)–(e) (page 19)

9.2.2 Intervals, domains and ranges

Let $a < b$. Then x lies in the **open interval** (a, b) if $a < x < b$; x lies in the **closed interval** $[a, b]$ if $a \leqslant x \leqslant b$. $(a, b]$ means $a < x \leqslant b$; $[a, b)$ means $a \leqslant x < b$.

The domain of a function may be explicitly described together with the function, or it may be implied by the equation used to define the function. The implied domain is the set of all real numbers for which the equation is defined. For example, the function

$$f(x) = \frac{1}{x^2 - 4}, \quad 4 \leqslant x \leqslant 5$$

has the explicitly defined domain $\{x : 4 \leqslant x \leqslant 5\} = [4, 5]$. On the other hand, the function

$$g(x) = \frac{1}{x^2 - 4}$$

had the implied domain $\{x : x \neq \pm 2\}$.

Another common type of implied domain is that used to avoid even roots of negative numbers. For example, the function $f(x) = \sqrt{x + 2}$ has the implied domain $\{x : x \geqslant -2\}$.

Example 2   *Finding the domain and range of a function*

Determine the domain and range of the function f defined by $f(x) = \sqrt{x - 1}$.

$\sqrt{x - 1}$ is not defined for $x - 1 < 0$, that is for $x < 1$, so we must have $x \geqslant 1$. Therefore, the domain is the interval $[1, \infty)$.
To find the range, we observe that $f(x)$ is never negative. Moreover, as x takes on the various values in the domain, $f(x)$ takes on all non-negative numbers. The range therefore is $[0, \infty)$.

Calculator details for the plots are in Appendix 2 on page 37.
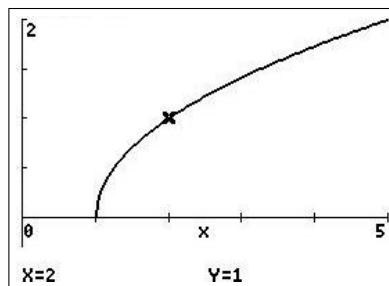
Example 3   *A function defined by more than one formula*

$$f(x) = \begin{cases} \sqrt{x - 1} & \text{if } x \geqslant 1 \\ 1 - x & \text{if } x < 1 \end{cases}$$

As f is defined for $x \geqslant 1$ and $x < 1$, the domain of the function is the set of all real numbers $(-\infty, \infty)$.

For $x \geqslant 1$, the function behaves as in the previous example. For $x < 1$, $1 - x$ is positive, and therefore the range of the function is the interval $[0, \infty)$. The graph of the function helps verify our conclusions.

A function defined in several parts, like this one, is called a *piecewise* function.
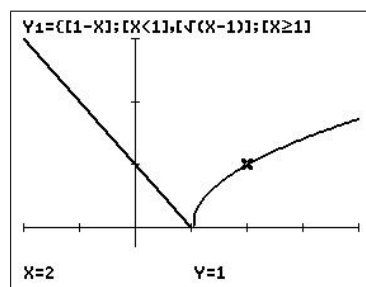
window $[-2, 4, 1] \times [0, 3, 1]$

Remember that the graph of the function $y = f(x)$ consists of all points $(x, y) = \big(x, f(x)\big)$, where x is the directed distance from the y axis and $y = f(x)$ the directed distance from the x axis.

4

By definition of a function f, there is exactly one y value for each x value in the domain of f. It follows that a vertical line can intersect the graph of a function at most once. This provides us with a convenient visual test for functions. For example, in the left-hand figure below, the graph is not that of a function because a vertical line intersects the graph twice.
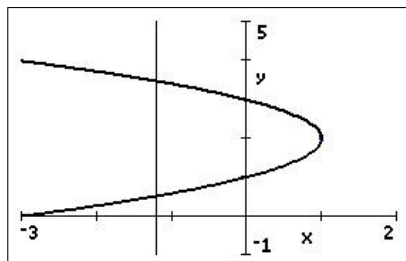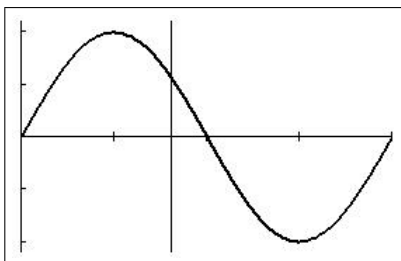
| Not a function of x | A function of x | A function of x |
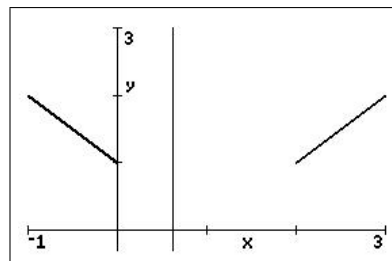
Vertical-line test for functions

Exercise 5 (page 20)

PTO

9.2.3 Basic transformations

Functional notation lends itself well to describing transformation of graphs in the plane. Some families of graphs all have the same basic shape. For example, consider the graph of $y = x^2$, as shown below left. Compare this graph with the other graphs; the graph of $y = x^2$ is shown as a thin dotted line in these.
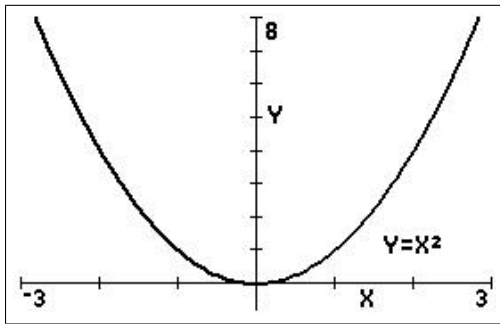
Original graph
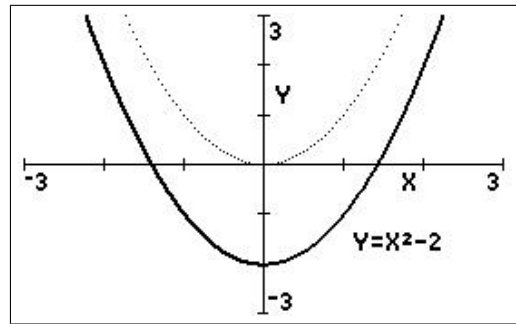
Vertical shift downward
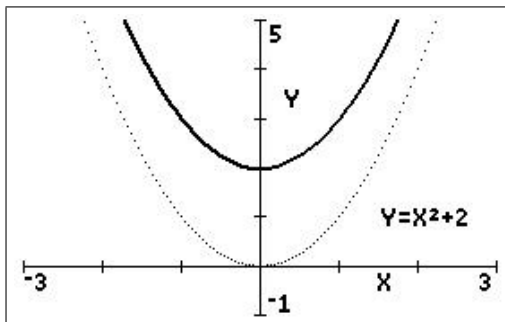
Vertical shift upwards
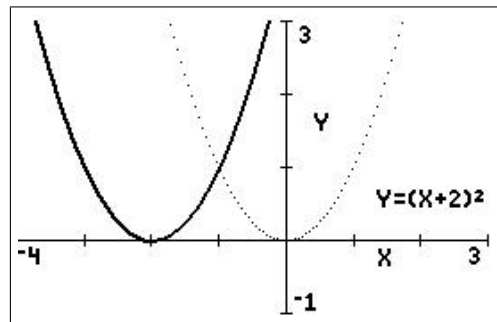
Horizontal shift to the left
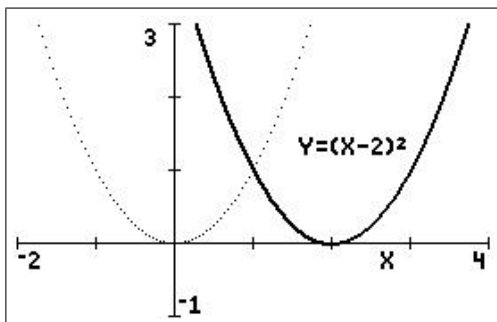
Horizontal shift to the right
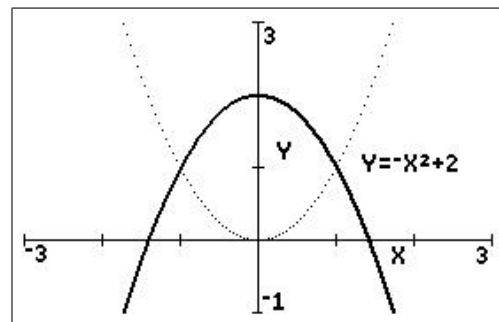
Reflection in x axis and vertical shift
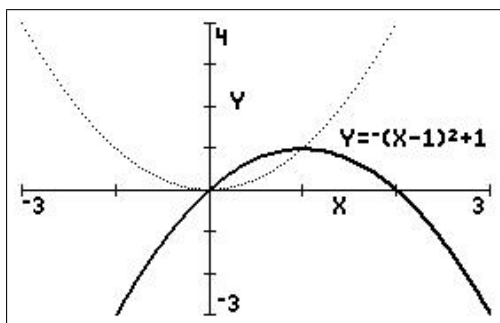
Reflection in x axis, vertical, horizontal shifts
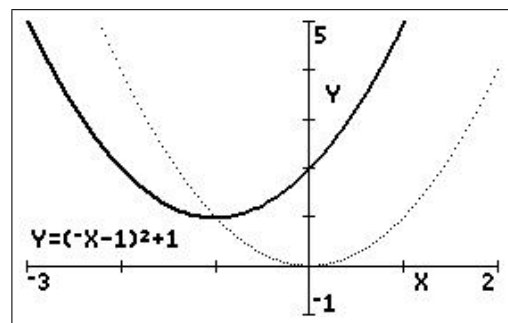
Vertical, horizontal shifts, reflection in y axis

Each of the graphs above is a **transformation** of the graph of $y = x^2$. The three basic types of transformations involved in these seven graphs are: (1) *horizontal shifts*; (2) *vertical shifts*; and (3) *reflections*.

Basic types of transformations $(c > 0)$

Original graph	$y = f(x)$
Horizontal shift c units to the right	$y = f(x-c)$
Horizontal shift c units to the left	$y = f(x+c)$
Vertical shift c units downward	$y = f(x)-c$
Vertical shift c units upward	$y = f(x)+c$
Reflection about the x axis	$y = -f(x)$
Reflection about the y axis	$y = f(-x)$

See *Appendix 1: Spinners* on page 32 for a fun way to explore function transformations.

9.2.4 Classifications and combinations of functions

The modern notion of a function is due to the efforts of many different mathematicians who lived in the seventeenth and eighteenth centuries. Of particular note was Leonhard Euler $(1707 - 1783)$, to whom we are indebted for the functional notation $y = f(x)$. The modern definition of a function was given by the German mathematician Peter Gustav Dirichlet $(1805 - 1859)$. Dirichlet made many contributions to Mathematics and, together with Cauchy, Riemann and Weierstrauss, developed much of the rigour present in Calculus today.

By the end of the eighteenth century, mathematicians and scientists had concluded that most real-world phenomena can be represented by mathematical models taken from a basic collection of functions called **elementary functions**. Elementary functions are divided into three categories: (1) *algebraic*; (2) *trigonometric*; and (3) *logarithmic and exponential*.

9.2.5 Power functions

Power functions are algebraic functions of the form $f(x) = ax^b$, where a and b are constants. The first few of these with b zero or a positive integer $(a = 1)$ are well-known: $f(x) = x^0 = 1$, a constant; $f(x) = x^1 = x$, a linear function; $f(x) = x^2$, a quadratic function; $f(x) = x^3$, a cubic function; and so on; all have domain $(-\infty, \infty)$. The figure below shows the graphs of these functions. The graphs of all of them, except the first, pass through the origin.
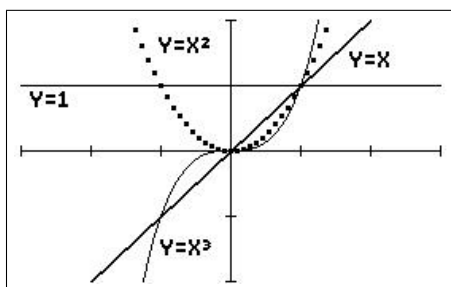
window $[-3, 3, 1] \times [-2, 2, 1]$

Power functions with exponent b an even integer are even functions (Section 9.2.9), with range $[0, \infty)$ (except x^0), those with b an odd integer are odd functions, with range $(-\infty, \infty)$.

If b is a fraction, say $b = c/d$, roots of numbers come into play: $f(x) = x^{\frac{c}{d}} = \sqrt[d]{x^c}$, that is $f(x)$ is the dth root of x^c.

For example, $f(x) = x^{\frac{1}{2}} = \sqrt{x}$, $f(x) = x^{\frac{2}{3}} = \sqrt[3]{x^2}$. The graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$.
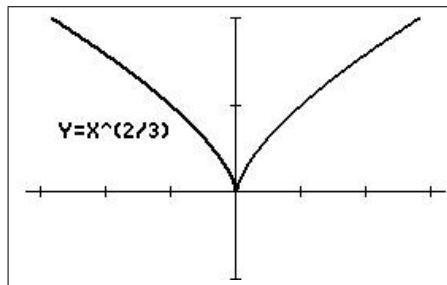
window $[-3.2, 3.2, 1] \times [-1, 2, 1]$

The graph of $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$ has a vertical point of inflection at $x = 0$.
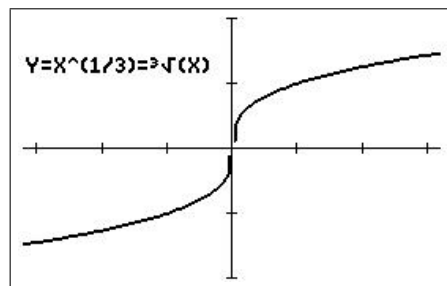
window $[-3.2, 3.2, 1] \times [-2, 2, 1]$

See Section 9.6.5.

9.2.6 Polynomial functions

The most common type of algebraic function is a **polynomial function**.

Definition   *polynomial function*

Let n be a non-negative integer. The function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function** of degree n. The numbers a_i, $i = 0, \ldots, n$ are called **coefficients**, with $a_n \neq 0$ the **leading coefficient** and a_0 the **constant term** of the polynomial function.

Remark: It is common practice to use subscript notation for coefficients of general polynomial functions but, for polynomial functions of low degree, we often use the following simpler forms.

Zeroth degree	$f(x) = a$	constant
First degree	$f(x) = ax + b$	linear
Second degree	$f(x) = ax^2 + bx + c$	quadratic
Third degree	$f(x) = ax^3 + bx^2 + cx + d$	cubic

See Sections 9.6.1 – 9.6.4.

Although the graph of a polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the left or right. Whether the graph eventually rises or falls is determined by the function's degree (odd or even) and by the sign of the leading coefficient, as indicated in the figures below.

Leading-coefficient test for polynomial functions

Note that this test only determines the ultimate behaviour of the graphs, i.e. as $x \to \pm\infty$.

$$a_n > 0 \qquad\qquad\qquad\qquad a_n < 0$$
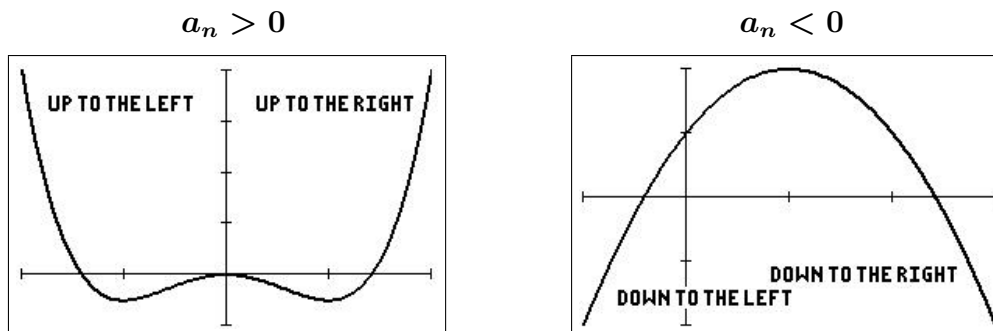
Polynomial functions of *even* degree

$$a_n > 0 \qquad\qquad\qquad\qquad a_n < 0$$
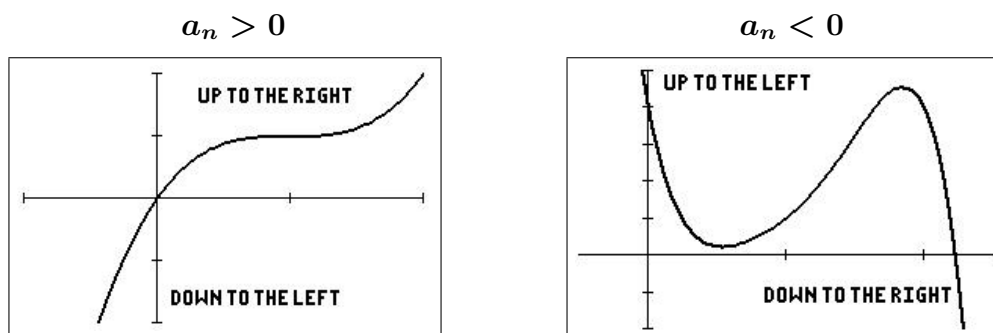
Polynomial functions of *odd* degree

9.2.7   Rational functions

Just as a rational number is the quotient of two integers, a **rational function** is the quotient of two polynomials. Specifically, a function f is rational if it has the form

$$f(x) = \frac{P(x)}{Q(x)}, \quad Q(x) \neq 0,$$

where P and Q are polynomials. See the separate activity *Let's Be Rational*[4] for an introduction to rational functions.

Power functions, polynomial functions and rational functions are examples of **algebraic functions**. An algebraic function is one that can be expressed in terms of finitely many sums, differences, multiples, quotients and radicals (roots) involving x^n. For example, the following functions are algebraic:

$$f(x) = \sqrt{x+1} \quad \text{and} \quad g(x) = x + \frac{1}{\sqrt{x+1}}.$$

Elementary functions that are not algebraic are called **transcendental**.

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[4]available at *www.YYY.*

9.2.8 Combinations of functions

Two functions can be combined in various ways to create new functions. For example, if

$$f(x) = 2x-3 \quad \text{and} \quad g(x) = x^2+1,$$

we can form the functions

$$
\begin{aligned}
f(x) + g(x) &= (2x-3) + (x^2+1) = x^2 + 2x + 2 && \text{sum} \\
f(x) - g(x) &= (2x-3) - (x^2+1) = -x^2 + 2x - 4 && \text{difference} \\
f(x)g(x) &= (2x-3)(x^2+1) = 2x^3 - 3x^2 + 2x - 3 && \text{product} \\
\frac{f(x)}{g(x)} &= \frac{2x-3}{x^2+1} && \text{quotient}
\end{aligned}
$$

Composite functions

Two functions can be combined in yet another way to form what is called a composite function.

Definition   *composite function*

Let f and g be functions such that the range of g is the domain of f. The function given by $f \circ g(x) = f\big(g(x)\big)$ is called the **composite** of f with g.

It is important to realise that the composite of f with g may not be equal to the composite of g with f. This is illustrated in the following example.

Example 4   *composite functions*

Given $f(x) = 2x-3$ and $g(x) = x^2+1$, find $f\big(g(x)\big)$ and $g\big(f(x)\big)$.

$f\big(g(x)\big) = 2\big(g(x)\big) - 3 = 2\big(x^2+1\big) - 3 = 2x^2 - 1.$

$g\big(f(x)\big) = \big(f(x)\big)^2 + 1 = (2x-3)^2 + 1 = 4x^2 - 12x + 10.$

Note: $f\big(g(x)\big) \neq g\big(f(x)\big)$, i.e. $f \circ g \neq g \circ f$.

Exercises 1(g), (h); 3(f)–(i) (page 19)

9.2.9 Even and odd functions

A function is even if its graph is symmetric with respect to the y axis; a function is odd if its graph is symmetric 'with respect to the origin'.

The algebraic equivalents, and a means for testing functions, are given in the following definition.

Definition   *even and odd functions*

The function $y = f(x)$ is *even* if $f(-x) = f(x)$.

The function $y = f(x)$ is *odd* if $f(-x) = -f(x)$.

Remark: Except for trivial cases such as $f(x) = 0$, the graph of a function cannot have symmetry with respect to the x axis, as it then fails the vertical-line test for the graph of a function.

Example 5   *even and odd functions*

Determine algebraically whether the following functions are even, odd or neither. Graph the function to confirm your answer.

   (a) $f(x)=x^3-x$.

      This function is odd because $f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3-x) = -f(x)$.
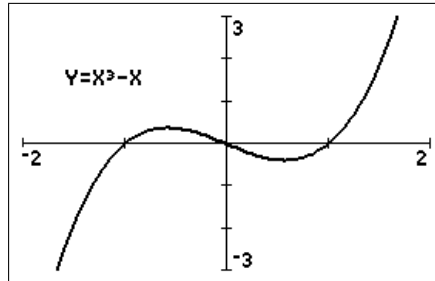
   (b) $f(x)=x^2+1$.

      This function is even because $f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$.
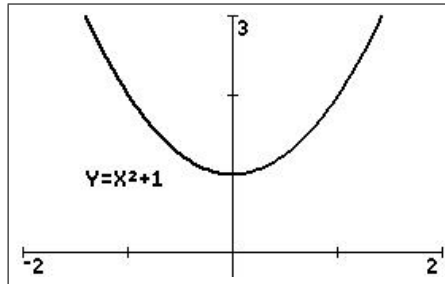
   (c) $f(x)=x+1$.

      This function is neither because $f(-x) = (-x) + 1 = -x + 1 \neq f(x)$ or $-f(x)$.
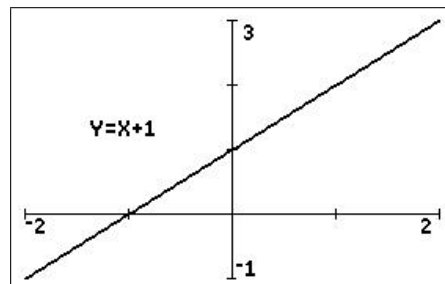
Most functions are neither even nor odd.

A function can always be split into an even part and an odd part:

$$f(x) \;=\; \frac{1}{2}\Big(f(x)+f(-x)\Big) \;+\; \frac{1}{2}\Big(f(x)-f(-x)\Big) \;=\; g(x) \;+\; h(x).$$

Then, $g(-x) = \frac{1}{2}(f(-x)+f(x)) = g(x)$,   so g is even, and

$$h(-x) = \tfrac{1}{2}(f(-x)-f(x)) = -\tfrac{1}{2}(f(x)-f(-x)) = -h(x), \;\; \text{so } h \text{ is odd.}$$

For example, if $f(x)=x+1$, then $g(x)=1$ and $h(x)=x$.

Exercise 6 (page 20)

9.2.10 Monotonic functions

$f(x)$ *increases* (strictly) *monotonically* in an interval (a, b) if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all pairs of numbers x_1, x_2 in (a, b). For example, $f(x) = \sqrt{x}$ is a (strictly) monotonically increasing function on the interval $[0, \infty)$.
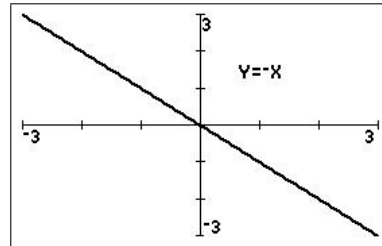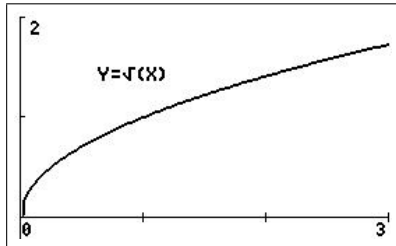
$f(x)$ *decreases* (strictly) *monotonically* in an interval (a, b) if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$. For example, $y = -x$ is a (strictly) monotonically decreasing function on the interval $(-\infty, \infty)$.

9.2.11 Absolute value

The absolute value of $g(x)$ or mod $g(x)$, written $|g(x)|$, is **defined** as

$$|g(x)| = \left\{ \begin{array}{rcl} g(x) & \text{if} & g(x) \geq 0 \\ -g(x) & \text{if} & g(x) < 0. \end{array} \right.$$

In graphing absolute-value functions, the key point is the x value at which $g(x) = 0$; the defintion of the function is different either side of this value.

Example 6   *Absolute value*

Sketch the graph of $f(x) = |x|$ for $-2 < x < 2$.

abs in the $\boxed{\text{math}}$ NUM menu is the calculator function for absolute value.

$f(x) = -x$ for $-2 < x < 0$ and $f(x) = x$ for $0 \leqslant x < 2$. The left-hand part of the graph of $f(x)$, i.e. $f(x) = -x$, is just the graph of $f(x) = x$ reflected in the x axis.

This provides a general method for plotting absolute values: plot the whole function, then reflect the part of the function below the x axis in the x axis.
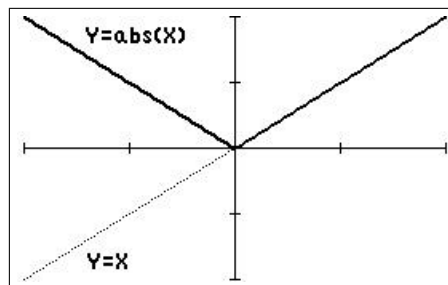
window $[-2, 2, 1] \times [-2, 2, 1]$

The graph here is called *piecewise linear* because each part of it is linear.

Example 7   *Absolute value*

Sketch the graph of $f(x) = |2x - 1|$ for $-2 < x < 2$.

The cross-over here is when $2x - 1 = 0$, i.e. $x = \frac{1}{2}$.

Then $f(x) = -(2x - 1)$ for $-2 < x < \frac{1}{2}$ and $f(x) = 2x - 1$ for $\frac{1}{2} \leqslant x < 2$.

The left-hand part of the graph, $x < \frac{1}{2}$, again is just the graph of $f(x) = 2x - 1$ reflected in the x axis.
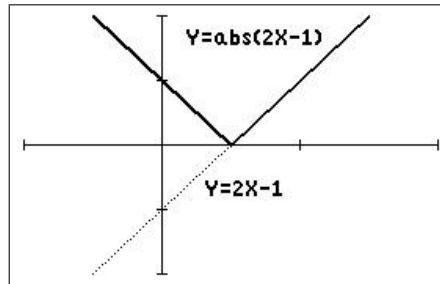
window $[-1, 2, 1] \times [-2, 2, 1]$

Exercise 4(a) (page 19)

9.2.12   Zeros of a function

When the graph of a function crosses the x axis (the value of the function is zero), we say that the function has a zero at that point. For example, the function $f(x) = x^2 - 4x + 3$ has a zero at $x = 1$, as $f(1) = 0$, and at $x = 3$, as $f(3) = 0$.
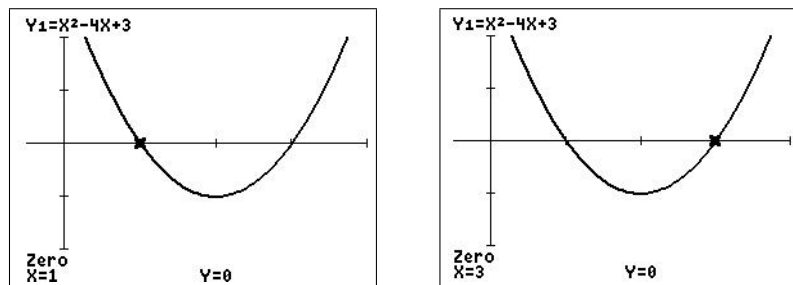
Zeros of a function

Algebraically, we find the zeros of a function by setting $f(x) = 0$ and solving for x. For the example here, we have

$$x^2 - 4x + 3 = 0.$$

$$\therefore (x - 1)(x - 3) = 0 \quad \text{factorise the quadratic.}$$

$$\therefore x = 1 \text{ or } 3.$$

Alternatively, use the quadratic formula to find the zeros.

9.3   Some useful and interesting functions

9.3.1   Exponential and logarithmic functions

The general exponential function (an elementary function) is of the form $f(x) = ab^x$, where a and b are constants. Compare these with power functions in Section 9.2.5.

Several exponential functions are shown in the figure below. All exponential functions pass through the point $(0, 1)$.
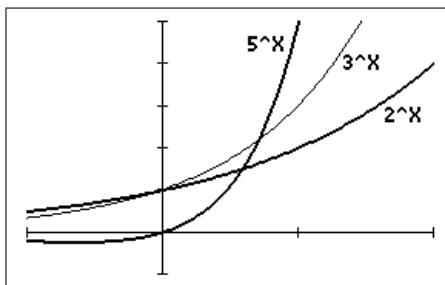
window $[-1, 2, 1] \times [-1, 5, 1]$

Exponential functions occur frequently in modelling. Unless the model involves a discrete process such as bacteria dividing, for which $f(x) = 2^x$ is an obvious choice, the exponential function $f(x) = ae^{bx}$ is often used, where $e = 2.7182818\ldots$. This function represents growth when b is positive, decay when b is negative.[5]
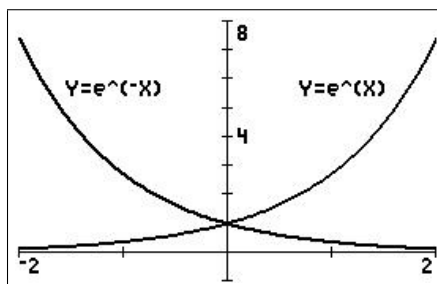
Logarithmic functions (elementary functions) occur sometimes in their own right but often because they are inverses of the corresponding exponential fuctions: e^x and $\ln(x)$ are inverses, so that
$$e^{\ln(x)} = x \quad \text{and} \quad \ln(e^x) = x.$$

The graphs of e^x and $\ln(x)$ are reflections of each other in the graph of $y = x$, as are all functions and their inverses.
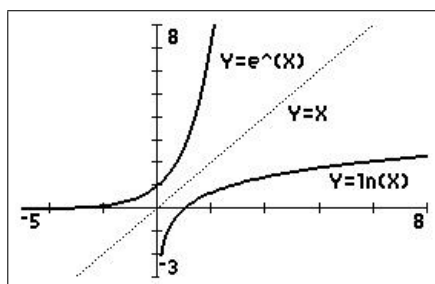
See Section 9.6.5.

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[5] $f(x) = e^x$ has a unique place in Calculus, being the only function which is its own derivative.

9.3.2   Greatest-integer function

The **greatest-integer function** or **floor function** $\lfloor x$ is defined to be the greatest integer less than or equal to x. If $x = 1$, then $\lfloor x = 1$; if $x = 1.2$, then $\lfloor x = 1$, etc. The graph of the greatest-integer function is montonic increasing and piecewise. The left-hand end of each interval is a closed circle (included), the right-hand end an open circle (not included).
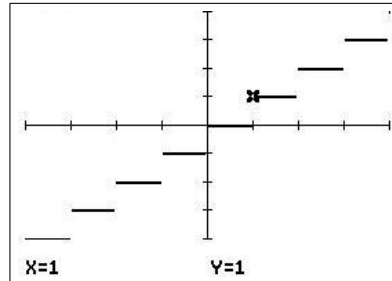
window $[-4, 4, 1] \times [-4, 4, 1]$

9.3.3   Least-integer function

The **least-integer function** or **ceiling function** $\lceil x$ is defined to be the least integer greater than or equal to x. If $x = 1$, then $\lceil x = 1$; if $x = 1.2$, then $\lceil x = 2$, etc. The graph of the least-integer function is also montonic increasing and piecewise. The right-hand end of each interval is a closed circle (included), the left-hand end an open circle (not included).
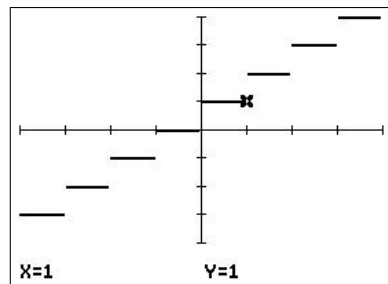
window $[-4, 4, 1] \times [-4, 4, 1]$

Examples of the greatest-integer function and least-integer function occur frequently in practice. In computing using integer arithmetic, the number 5.67 would be truncated to 5, an example of the greatest-integer function, The height of a staircase is an example of the least-integer function, and so are postal charges: e.g. letters up to $20\,\mathrm{g}$ cost \$1, letters over $20\,\mathrm{g}$ and up to $50\,\mathrm{g}$ cost \$1.50, etc. Charges for parking in parking stations are also least-integer functions.

PTO

Example 8

Sketch the graph of $f(x) = x \lfloor x$ for $-2 < x < 2$.

$\lfloor x$ is constant between integers, so that $f(x)$ is a series of straight-line segments of different slopes. The left-hand end of each interval is a closed circle (included), the right-hand end an open circle (not included). This is another piecewise-linear function.

x	$\lfloor x$	$x \lfloor x$
-2	-2	4
-1.5	-2	3
-1	-1	1
-0.5	-1	0.5
0	0	0
0.5	0	0
1	1	1
1.5	1	1.5
2	2	4
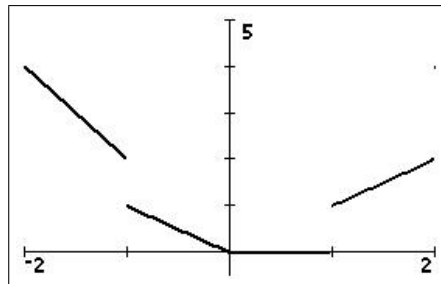
9.3.4 Periodic functions

Example 9

Sketch the graph of $f(x) = x - \lfloor x$ for $-3 < x < 3$.

x	$\lfloor x$	$x - \lfloor x$
-3	-3	0
-2.75	-3	0.25
-2.5	-3	0.5
-2.25	-3	0.75
-2	-2	0
-1.75	-2	0.25
-1.5	-2	0.5
-1.25	-2	0.75
-1	-1	0
-0.75	-1	0.25
-0.5	-1	0.5
-0.25	-1	0.75
0	0	0
$\vdots$	$\vdots$	$\vdots$
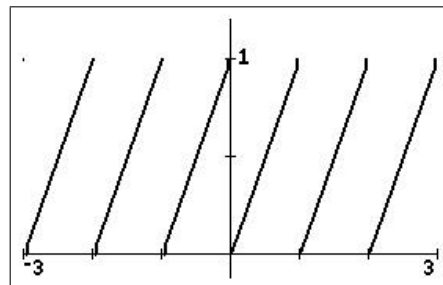
This is an example of a **periodic function**, one that repeats itself at regular intervals. Mathematically, $f(x+a) = f(x)$, where the constant a is called the period. The function $x - \lfloor x$ has period 1. This is also a piecewise-linear function.

The trigonometric functions $\sin(x)$, $\cos(x)$ and $\tan(x)$ (elementary functions) are periodic functions with period 2π. $\tan(x)$ has vertical asymptotes at odd multiples of $\pi/2$.
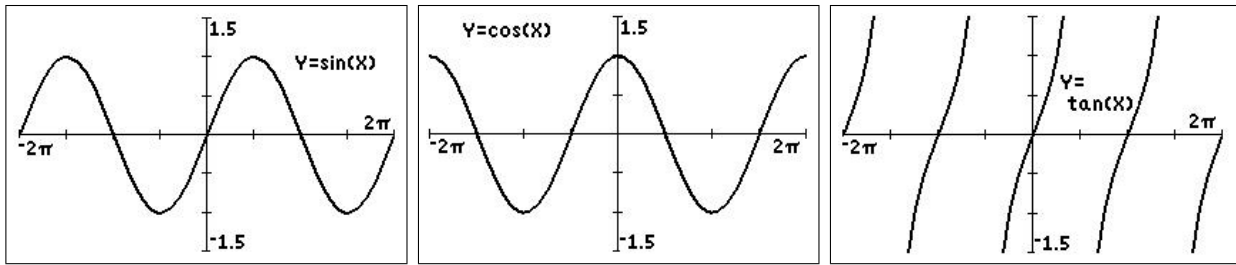
See Sections 9.6.6, 9.6.7 and 9.6.10.

Example 10

Sketch the graph of $f(x)$ for $-3 < x < 3$, where $f(x) = \begin{cases} x & 0 \le x < 1 \\ f(x+1) & \text{for all } x. \end{cases}$

x	$f(x)$
$\vdots$	$\vdots$
-1	0
-0.75	0.25
-0.5	0.5
-0.25	0.75
0	0
0.25	0.25
0.5	0.5
0.75	0.75
1	0
1.25	0.25
1.5	0.5
1.75	0.75
$\vdots$	$\vdots$
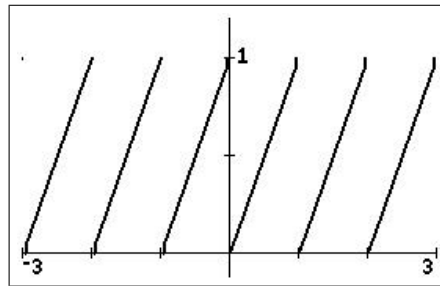
This is the same as Example 9, which shows that there may be more than one way of expressing a function.

Exercise 4(b) (page 19)

9.3.5   Heaviside function

The Heaviside (step) function is piecewise (and monotonic) like the greatest-integer and least-integer functions but takes only two values, 0 and 1. It is useful in modelling, for example, electrical circuits because it is a 'switch-on' or step function: it switches on, changes from 0 to 1, at the point where its argument is 0. With a minus sign in front, it becomes a 'switch-off' function.

The Heaviside function is defined by

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}.$$

Note that $H(0)$ is undefined. This is another piecewise-linear function.

Example 11

1. $f(x) = H(x)$, the basic Heaviside function: switches on at $x=0$.
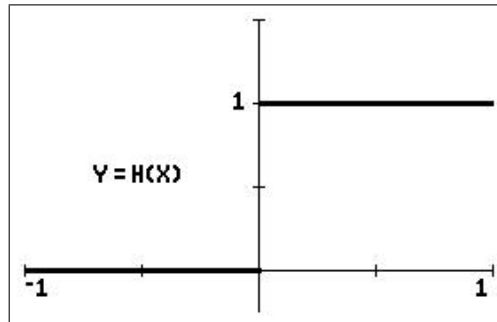
2. $f(x) = H(x-1)$: switches on at $x=1$.
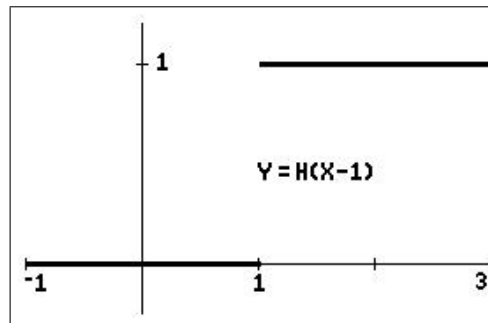
3. $f(x) = xH(x-1) + 1$: jumps up 1 unit at $x=1$, then increases linearly as $y=x+1$.
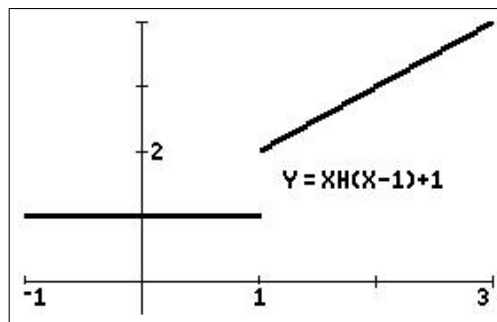
4. $f(x) = H(x) - H(x-1)$: switches on at $x=0$, then switches off at $x=1$.
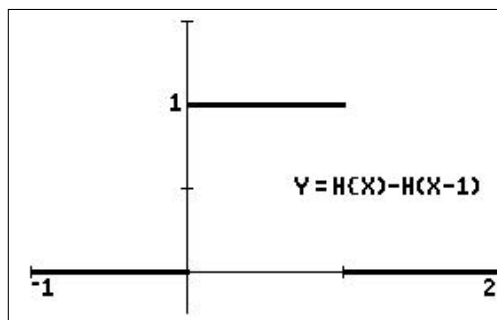
Exercise 7 (page 20)

9.4   Exercises

Solutions are in the next section.

1. If $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{2x+5}{4x-3}$, find

(a) $f(2)$

(b) $f(0)$

(c) $f(x^2)$

(d) $g(t)$

(e) $g(s)$

(f) $f\big(\tan(\theta)\big)$

(g) $f\big(g(x)\big)$

(h) $g\big(f(x)\big)$

2. Sketch the graph of the following functions by hand or on your calculator. If the domain is not stated, assume the maximum possible domain.

(a) $y = \dfrac{x}{x-1}$

(b) $y = |x-1| + 1$

(c) $y = \dfrac{x}{x^2+1}$

(d) $y = \lfloor x,\ -3 \leqslant x \leqslant 3$

(e) $y = x + \lfloor x,\ -2 \leqslant x \leqslant 2$

(f) $y = \lfloor x^2,\ -2 \leqslant x \leqslant 2$

(g) $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$

(h) $y = x|x|$

3. If $f(x) = |x|$ and $g(x) = x^2$, find

(a) $f(-1)$

(b) $g(2)$

(c) $f(x^2)$

(d) $g\big(\cos(t)\big)$

(e) $g\left(\dfrac{1}{x}\right)$

(f) $f\big(g(x)\big)$

(g) $g\big(f(x)\big)$

(h) $f(x+y)$

(i) $f\big(g(x+y)\big)$

4. Sketch the following functions by hand. Check with your calculator.

(a) $f(x) = |x+1| + |2x-1|;\ -2 < x < 2.$

Hint: Consider the x values at which each term in the function equals 0. Either use these values to draw up a table of function values or work out the expression for the function in each of three intervals of the x axis.

(b) $f(x) = x^2,\ 0 \leqslant x < 1$ and $f(x+1) = f(x)$ for all x. Plot the function for $-2 < x < 2$.

PTO

5. Find the range and domain of the following functions. Sketch their graphs, using your calculator if necessary.

(a) $f(x) = (x-1)^{\frac{1}{2}}$

(b) $f(x) = (1-x)^{\frac{1}{2}}$

(g) $f(x) = \dfrac{1}{|x|}$

(c) $f(x) = x^2$

(h) $f(x) = |x-2|$

(d) $f(x) = 4-x^2$

(i) $f(x) = \dfrac{|x|}{x}$

(e) $f(x) = (9-x^2)^{\frac{1}{2}}$

(j) $f(x) = (x^2-4)^{\frac{1}{2}}$.

(f) $f(x) = (25-x^2)^{\frac{1}{2}}$

6. Determine algebraically if the following functions are even, odd or neither. Confirm your answer by graphing the function, using your calculator if necessary.

(a) $f(x) = 4-x^2$

(d) $f(x) = 1+x^3$

(b) $f(x) = x^3$

(e) $f(x) = 4x-x^2$

(c) $f(x) = x(4-x^2)$

(f) $f(x) = x^{\frac{2}{3}}$. Hint: $x^{\frac{a}{b}} = \left(x^a\right)^{\frac{1}{b}}$

7. (a) Draw the following function on $[0, 4]$:

$$f(x) = \begin{cases} 0 & 0 \leqslant x < 1 \\ 1 & 1 \leqslant x < 2 \\ 2 & 2 \leqslant x < 3 \\ 0 & 3 \leqslant x \leqslant 4 \end{cases}.$$

(b) Write down a formula for the function in (a) in terms of Heaviside functions.

9.5   Solutions to the exercises

1. If $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{2x+5}{4x-3}$,

(a) $f(2) = 3$

(b) $f(0) = -1$

(c) $f(x^2) = \dfrac{x^2+1}{x^2-1}$

(d) $g(t) = \dfrac{2t+5}{4t-3}$

(e) $g(s) = \dfrac{2s+5}{4s-3}$

(f) $f\big(\tan(\theta)\big) = \dfrac{\tan(\theta)+1}{\tan(\theta)-1}$

(g) More complicated!

$$
\begin{aligned}
f\big(g(x)\big) &= f\left(\frac{2x+5}{4x-3}\right)\\[2mm]
&= \frac{\left(\dfrac{2x+5}{4x-3}\right)+1}{\left(\dfrac{2x+5}{4x-3}\right)-1} \times \frac{4x-3}{4x-3}\\[2mm]
&= \frac{2x+5+(4x-3)}{2x+5-(4x-3)}\\[2mm]
&= \frac{6x+2}{-2x+8}\\[2mm]
&= \frac{3x+1}{-x+4}
\end{aligned}
$$

(h) Again, a bit of algebra.

$$
\begin{aligned}
g\big(f(x)\big) &= g\left(\frac{x+1}{x-1}\right)\\[2mm]
&= \frac{2\left(\dfrac{x+1}{x-1}\right)+5}{4\left(\dfrac{x+1}{x-1}\right)-3} \times \frac{x-1}{x-1}\\[2mm]
&= \frac{2(x+1)+5(x-1)}{4(x+1)-3(x-1)}\\[2mm]
&= \frac{7x-3}{x+7}.
\end{aligned}
$$

2. Sketch the graph of the following functions by hand or on your calculator. If the domain is not stated, assume the maximum possible domain.

(a) $y = \dfrac{x}{x-1}$ Y₁ = X/(X−1)
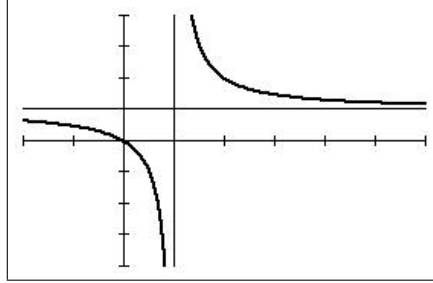
window $[-2, 6, 1] \times [-4, 4, 1]$

Domain all x except $x = 1$. Note the horizontal asymptote at $y = 1$ and the vertical asymptote at $x = 1$.

(b) $y = |x-1| + 1$ Y₁ = abs(X−1)+1   *abs* is in the $\boxed{\text{math}}$ NUM menu
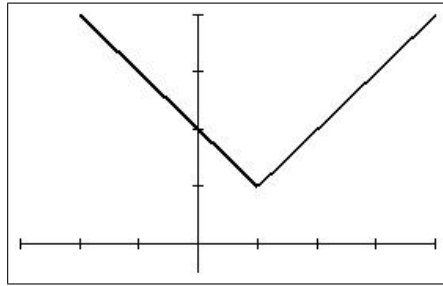
window $[-3, 4, 1] \times [-0.5, 4, 1]$

Domain all x.

(c) $y = \dfrac{x}{x^2+1}$ Y₁ = X/(X²+1)
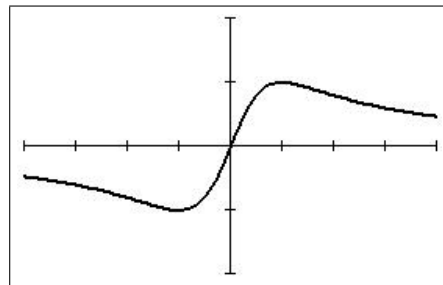
window $[-4, 4, 1] \times [-1, 1, 0.5]$

Domain all x.

22

(d) $y = \lfloor x, \ -3 \leqslant x \leqslant 3$　　　$Y_1 = \mathrm{int}(X)$　　　*int* is in the $\boxed{\mathrm{math}}$ NUM menu
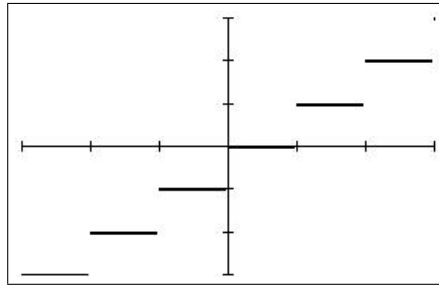
window $[-3, 3, 1] \times [-3, 3, 1]$

Domain all x. The left-hand end of each line interval is included (closed circle), the right-hand end is not (open circle).

(e) $y = x + \lfloor x, \ -2 \leqslant x \leqslant 2$　　　$Y_1 = X + \mathrm{int}(X)$
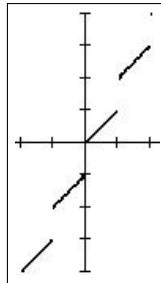
window $[-2, 2, 1] \times [-4, 4, 1]$

Domain all x. The left-hand end of each line interval is included (closed circle), the right-hand end is not (open circle).

(f) $y = \lfloor x^2, \ -2 \leqslant x \leqslant 2$　　　$Y_1 = \mathrm{int}(X^2)$
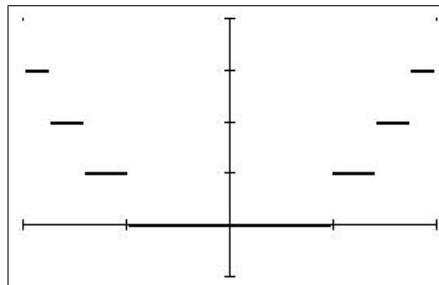
window $[-2, 2, 1] \times [-1, 4, 1]$

Domain all x. The end of each line interval closer to the y axis is included (closed circle), the other end is not (open circle).

(g) $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$ just plot $Y_1 = (X^2-1)/(X-1)$; trace to $X = 1$.


```
Y1=(X²-1)/(X-1)




X=1              Y=
```
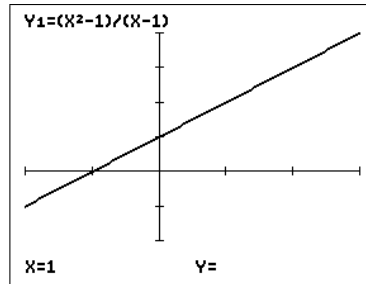
window $[-2, 3, 1] \times [-2, 4, 1]$

Domain all x. The first part of the function has a hole discontinuity at $x = 1$ (no Y value at $X = 1$); the second part 'fills in' that hole to make the function a continuous straight line.

(h) $y = x|x|$ $Y_1 = X \, \text{abs}(X)$
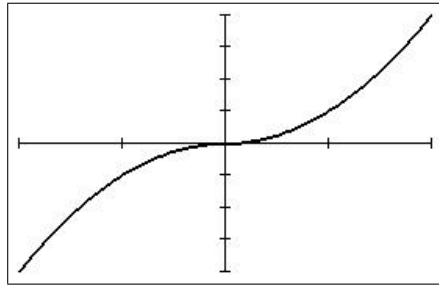
window $[-2, 2, 1] \times [-4, 4, 1]$

Domain all x. The part of the curve with $x \geqslant 0$ is just the graph of $y = x^2$; the part with $x < 0$ is the graph of $y = -x^2$, that is the graph of $y = x^2$ reflected in the x axis.

3. If $f(x) = |x|$ and $g(x) = x^2$,

(a) $f(-1) = 1$

(b) $g(2) = 4$

(c) $f(x^2) = x^2$

(d) $g\big(\cos(t)\big) = \cos^2(t)$

(e) $g\left(\dfrac{1}{x}\right) = \dfrac{1}{x^2}$

(f) $f\big(g(x)\big) = f(x^2) = |x^2| = x^2$

(g) $g\big(f(x)\big) = g(|x|) = |x|^2 = x^2$

here, $f \circ g = g \circ f$

(h) $f(x+y) = |x+y|$

(i) $f\big(g(x+y)\big) = |x+y|^2 = (x+y)^2$

4. Sketch the following functions. Check with your calculator.

(a) $f(x) = |x+1| + |2x-1|;\quad -2 < x < 2$ $Y_1 = \mathrm{abs}(X+1) + \mathrm{abs}(2X-1)$

Algebraically: the first term is 0 at $x = -1$, the second at $x = \frac{1}{2}$. Therefore, divide the x axis into three intervals: $-2 < x < -1;\ -1 < x < \frac{1}{2};$ and $\frac{1}{2} < x < 2$.

For $-2 < x < -1$, $|x+1| = -x-1$ and $|2x-1| = -2x+1$, so that $f(x) = -x-1-2x+1 = -3x$.

For $-1 < x < \frac{1}{2}$, $|x+1| = x+1$ and $|2x-1| = -2x+1$, so that $f(x) = x+1-2x+1 = -x+2$.

For $\frac{1}{2} < x < 2$, $|x+1| = x+1$ and $|2x-1| = 2x-1$, so that $f(x) = x+1+2x-1 = 3x$.

Check by plotting these three functions against the plot of $Y_1 = \mathrm{abs}(X+1) + \mathrm{abs}(2X-1)$.
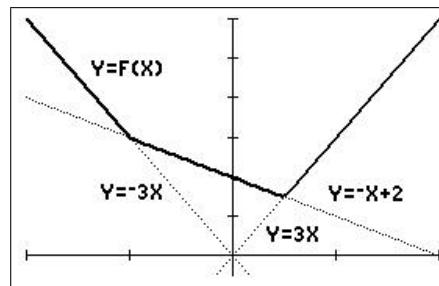
window $[-2, 2, 1] \times [-0.5, 6, 1]$

Domain all x. A piecewise-linear function.

(b) $f(x) = x^2,\ 0 \leqslant x < 1$ and $f(x+1) = f(x)$ for all x. Plot the function for $-2 \leqslant x < 2$.

Plot the basic function $y = x^2$ on $0 \leqslant x < 1$, then repeat the graph at x intervals of 1.
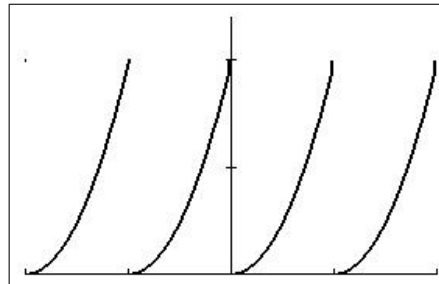
window $[-2, 2, 1] \times [0, 1.2, 0.5]$

Domain all x. This is a periodic function of period 1. The left-hand end of each part of the graph is included (closed circle), the right-hand end is not (open circle). This is a piecewise-quadratic function.

You could also use the PERIODIC/PERIODCE program: see page 40: $Y_1 = X^2$.

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5. Find the range and domain of the following functions. Sketch their graphs, using your calculator when necessary.

(a) $f(x) = (x-1)^{\frac{1}{2}}$ Y₁ = $\sqrt{(\text{X}-1)}$ or Y₁ = (X−1)^(1/2)

Domain $[1, \infty)$; range $[0, \infty)$.
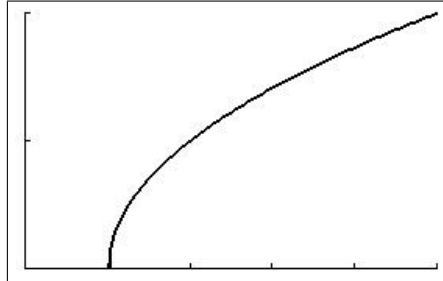
window $[0, 5, 1] \times [0, 2, 1]$

(b) $f(x) = (1-x)^{\frac{1}{2}}$ Y₁ = $\sqrt{(1-\text{X})}$

Domain $(-\infty, 1]$; range $[0, \infty)$.
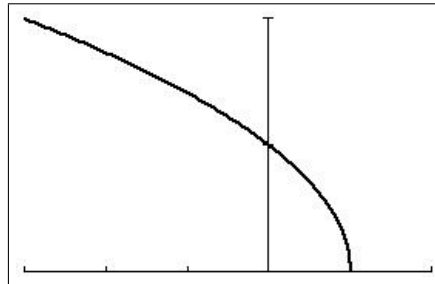
window $[-3, 2, 1] \times [0, 2, 1]$

(c) $f(x) = x^2$ Y₁ = X²

The basic parabola. Domain $(-\infty, \infty)$; range $[0, \infty)$.

window $[-3, 3, 1] \times [0, 10, 1]$

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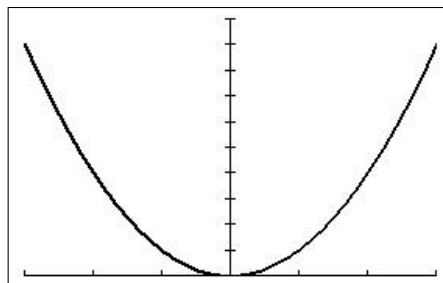
(d) $f(x) = 4-x^2$ $Y_1 = 4-X^2$

A head-up parabola. Domain $(-\infty, \infty)$; range $(-\infty, 4]$.
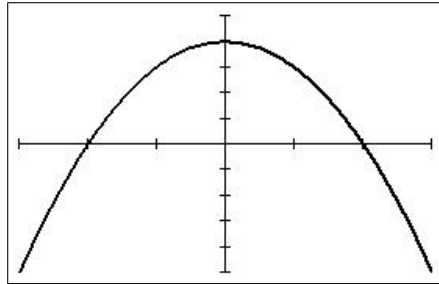
window $[-3, 3, 1] \times [-5, 5, 1]$

(e) $f(x) = (9-x^2)^{\frac{1}{2}}$ $Y_1 = \sqrt{(9-X^2)}$

A semi-circle, centre the origin, radius 3. Domain $[-3, 3]$; range $[0, 3]$.
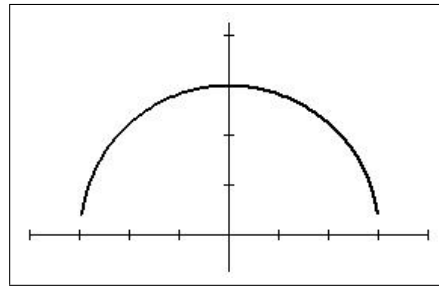
window $[-4, 4, 1] \times [0, 4, 1]$, then $\boxed{\text{zoom}}$ $\boxed{5}$

Note that the calculator does not draw the whole semi-circle; the near-vertical parts near the x axis are left out. You have to use parametric equations to have it drawn properly.

(f) $f(x) = (25-x^2)^{\frac{1}{2}}$ $Y_1 = \sqrt{(25-X^2)}$

A semi-circle, centre the origin, radius 5. Domain $[-5, 5]$; range $[0, 5]$.
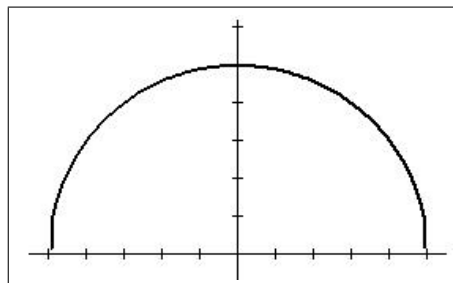
window $[-5, 5, 1] \times [0, 6, 1]$, then $\boxed{\text{zoom}}$ $\boxed{5}$

Here, the near-vertical parts near the x axis are mostly drawn in but not very accurately.

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(g) $f(x) = \dfrac{1}{|x|}$ $Y_1 = 1/\text{abs}(X)$

Domain $(-\infty, 0)$, $(0, \infty)$; range $(0, \infty)$.
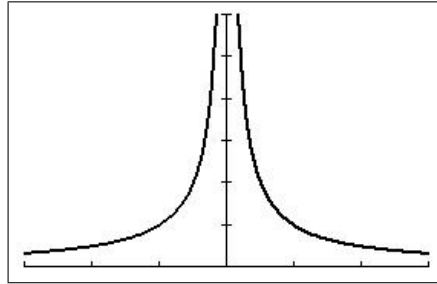
window $[-3, 3, 1] \times [0, 6, 1]$

Vertical asymptote at $x = 0$.

(h) $f(x) = |x-2|$ $Y_1 = \text{abs}(X-2)$

Domain $(-\infty, \infty)$; range $[0, \infty)$.
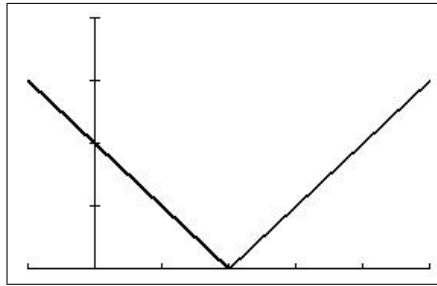
window $[-1, 5, 1] \times [0, 4, 1]$

(i) $f(x) = \dfrac{|x|}{x}$ $Y_1 = \text{abs}(X)/X$

Domain $(-\infty, 0)$, $(0, \infty)$; range $-1, 1$.

window $[-2, 2, 1] \times [-2, 2, 1]$

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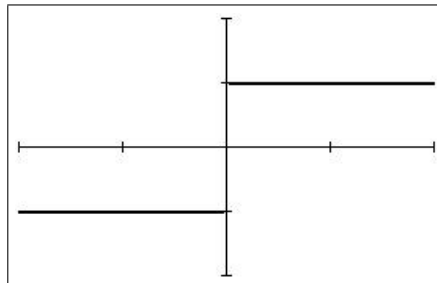
(j) $f(x) = (x^2 - 4)^{\frac{1}{2}}$ Y₁ = $\sqrt{(X^2 - 4)}$

Domain $(-\infty, -2], [2, \infty)$; range $[0, \infty)$.
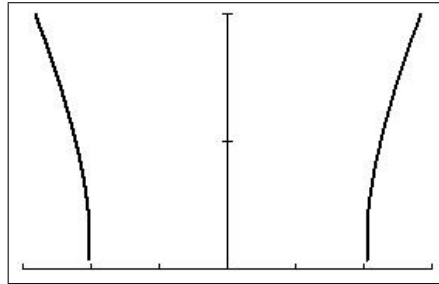
window $[-3, 3, 1] \times [0, 2, 1]$

See (f) above.

6. Determine algebraically if the following functions are even, odd or neither. Confirm your answer by graphing the function, using your calculator if necessary.

(a) $f(x) = 4 - x^2$ Y₁ = 4−X²

$f(-x) = 4 - (-x)^2 = 4 - x^2 = f(x)$. An even function.
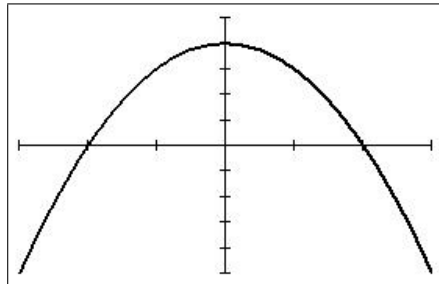
window $[-3, 3, 1] \times [-5, 5, 1]$

(b) $f(x) = x^3$ Y₁ = X^3 or Y₁ = X³   cubed is $\boxed{\text{math}}$ $\boxed{3}$

$f(-x) = (-x)^3 = -x^3 = -f(x)$. An odd function.
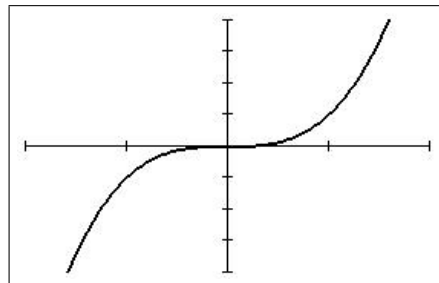
window $[-2, 2, 1] \times [-4, 4, 1]$

(c) $f(x) = x(4-x^2)$ $Y_1 = X(4-X^2)$

$f(-x) = (-x)(4-(-x)^2) = -x(4-x^2) = -f(x)$. An odd function.
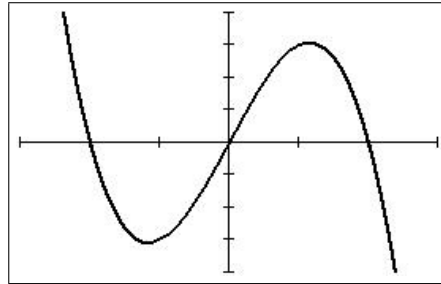
window $[-3, 3, 1] \times [-4, 4, 1]$

(d) $f(x) = 1 + x^3$ $Y_1 = 1+X^3$

$f(-x) = 1 + (-x)^3 = 1 - x^3 \neq f(x)$ or $-f(x)$. Neither even nor odd.
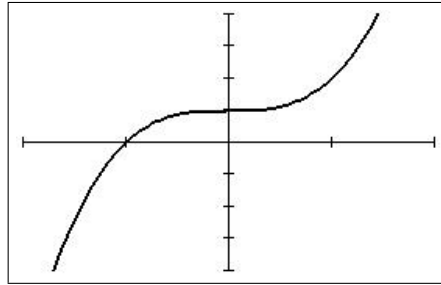
window $[-2, 2, 1] \times [-4, 4, 1]$

(e) $f(x) = 4x - x^2$ $Y_1 = 4X-X^2$

$f(-x) = 4(-x)-(-x)^2 = -4x - x^2 \neq f(x)$ or $-f(x)$. Neither even nor odd.
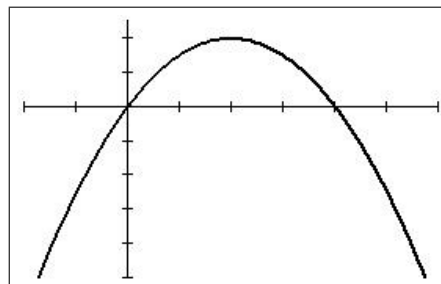
window $[-2, 6, 1] \times [-10, 5, 2]$

(f) $f(x) = x^{\frac{2}{3}}$ $Y_1 = X^\wedge(2/3)$

Write f as $f(x) = (x^2)^{\frac{1}{3}}$.

$f(-x) = \left((-x)^2\right)^{\frac{1}{3}} = (x^2)^{\frac{1}{3}} = x^{\frac{2}{3}} = f(x)$. An even function.
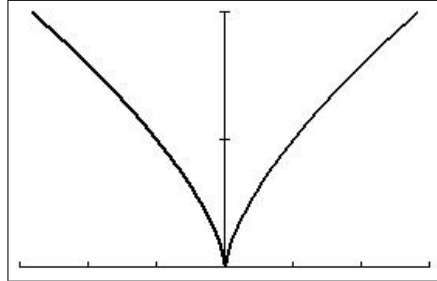
window $[-3, 3, 1] \times [0, 2, 1]$

7. (a) Draw the following function on $[0, 4]$

$$f(x) = \begin{cases} 0 & 0 \leqslant x < 1 \\ 1 & 1 \leqslant x < 2 \\ 2 & 2 \leqslant x < 3 \\ 0 & 3 \leqslant x \leqslant 4 \end{cases}.$$

X=2.5   Y=2
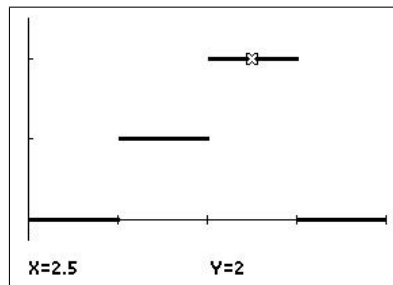
window $[0, 4, 1] \times [-0.25, 2.5, 1]$

(b) Write down a formula for the function in (a) in terms of Heaviside functions.

$$f(x) = H(x-1) + H(x-2) - 2H(x-3).$$

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9.6 Appendix 1: Spinners

Explore various types of functions with spinners[6] (here programs) on a TI-84 or CE. Spinners allow you to vary the function parameters on screen, by pressing a key, and see the result replotted immediately. All programs contain the necessary information (in HELP) on how to vary the parameters.

9.6.1 Quadratic functions

$Y = AX^2 + BX + C$. The program SPCPQUAD/SPQUADCE plots the quadratic function and the real zeros (if they exist; below left) or the complex zeros (taking the X axis as the real axis and the Y axis as the imaginary axis; below right). It also displays the discriminant on the screen.
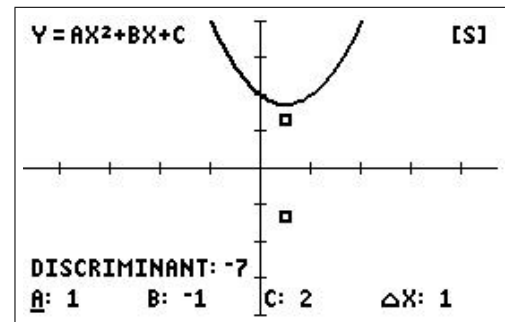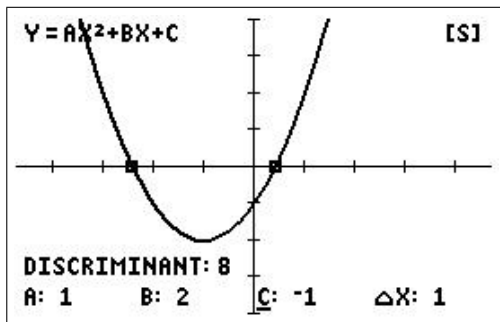
Values of A, B and C are changed by pressing arrow keys. The new graph is immediately redrawn.

9.6.2 Cubic functions

$Y = AX^3 + BX^2 + CX + D$. The program SPCUBIC/SPCBICCE plots the cubic function (below left) and has the option to plot its derivative (below right).

---
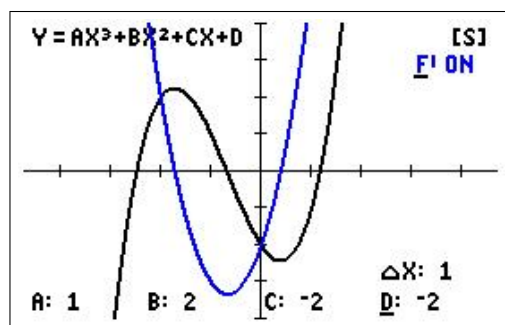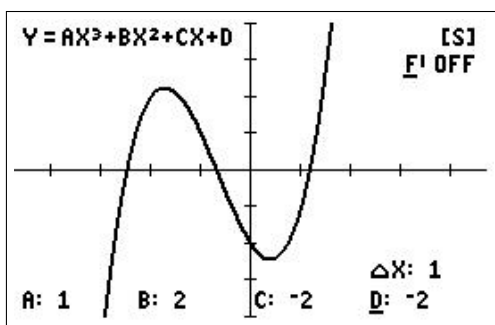
[6]From a feature in an earlier version of Excel used by Ed Staples and Margie Smith in SpinnaMaths to explore the effects of varying parameters in a variety of applications; sadly, SpinnaMaths is no longer available because of subsequent 'upgrades' to Excel.

The next two are for the real function enthusiast. You may need to edit the program and change the viewing window if you change the parameters by more than a couple.

9.6.3   Quartic functtions

$Y = AX^4 + BX^3 + CX^2 + DX + E$. The program SPQUART/SPQRTCE plots the quartic function (below left) and has the option to plot its derivative (below right).
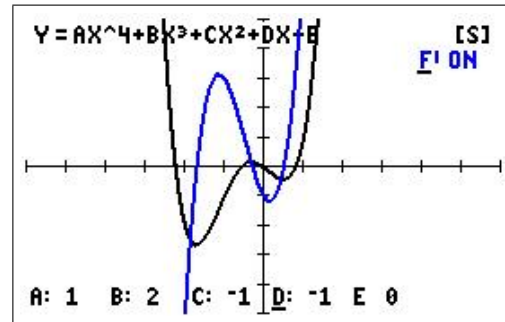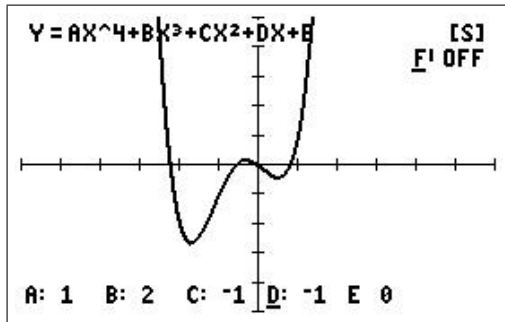
9.6.4   Quintic functions

$Y = AX^5 + BX^4 + CX^3 + DX^2 + EX + F$. The program SPQUINT/SPQNTCE plots the quintic function (below left) and has the option to plot its derivative (below right).
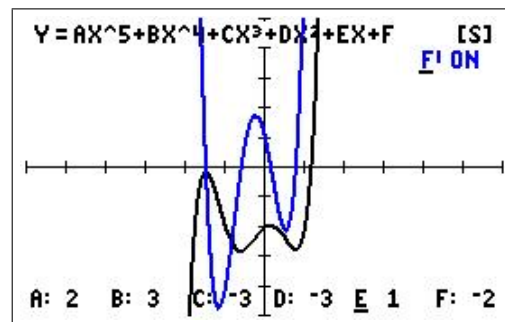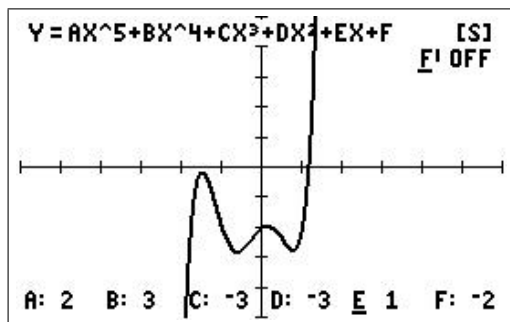
9.6.5   Power, exponential and logarithmic functions

The program SPPWEXLG/SPPELCE plots power functions (below left), exponential functions (below centre) and logarithmic functions to any base (below right).

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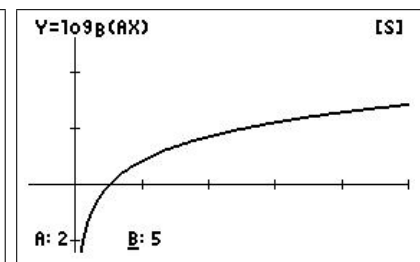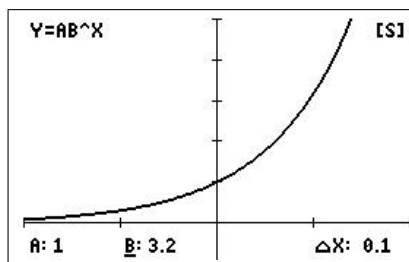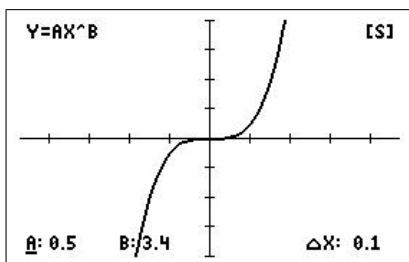
9.6.6   Trigonometric functions

The program SPTRIGF/SPTRIGCE plots the trigonometric functions sin (below left), cos and tan, their reciprocals cosec (below centre), sec and cot and their inverses $\sin^{-1}$ (arcsin; below right), $\cos^{-1}$ (arccos) and $\tan^{-1}$ (arctan).
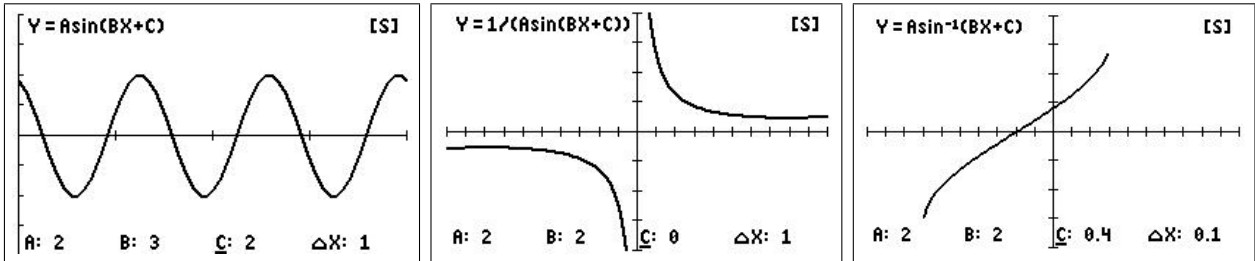
9.6.7   Sum of two sine functions

$Y = \sin(2\pi X) + A\sin(2\pi BX + C)$. The program SPSUMSIN/SPSSINCE plots the sum of two sin functions with different arguments.
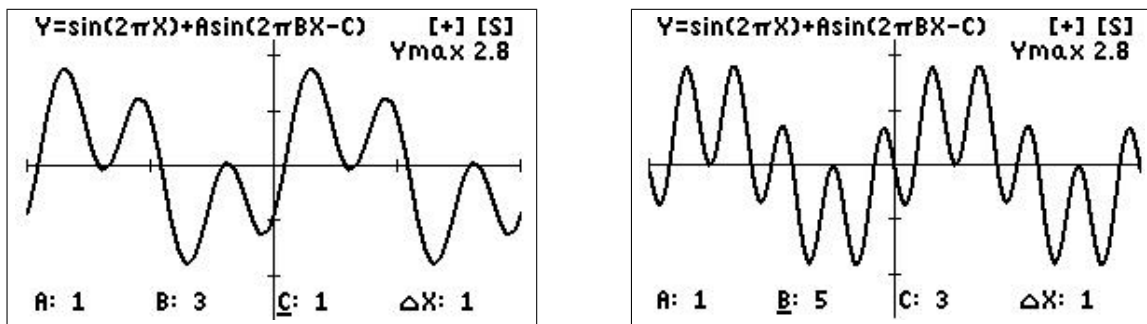
9.6.8   Graph your own function

The program SPGRAPH/SPGRPHCE plots whatever function you put in Y_1 with up to three parameters A, B and C (below left). There is also the option to plot its derivative (below right).
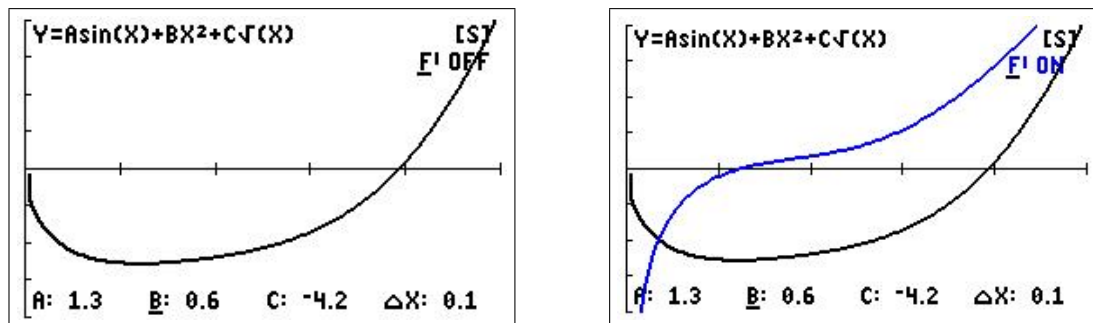
9.6.9   Hyperbolic functions

The program SHYPBFN/SPHYPFCE plots the hyperbolic functions sinh (below left), cosh and tanh, and their inverses $\sinh^{-1}$ (below right), $\cosh^{-1}$ and $\tanh^{-1}$.
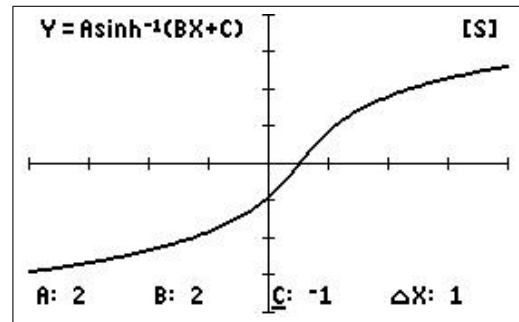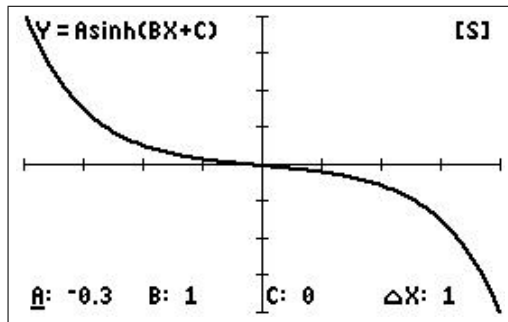
9.6.10   Simple harmonic motion

$Y = A\cos(KX) + B\sin(KX)$. The program SPSHMA/SPSHMACE plots the displacement Y of a body undergoing simple harmonic motion (below left), with the option of also plotting its velocity V (below right).
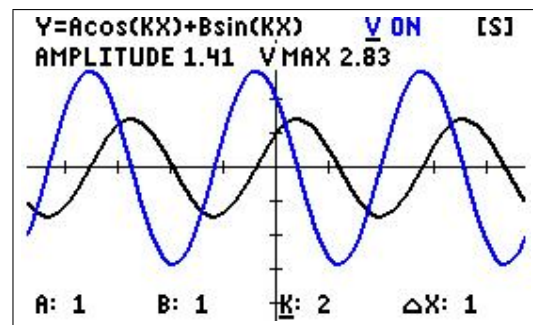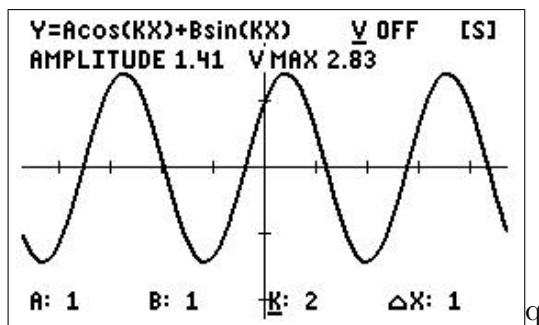
The program SPSHMB/SPSHMACB has the additional feature of a particle undergoing vertical simple harmonic motion concurrent with the plotting of the displacement graph (below left). Both the graph and particle position can be traced after the plot to show corresponding points on the graph and particle trajectory (below centre and right). The velocity of the particle, together with the particle moving along its vertical trajectory, can also be plotted.
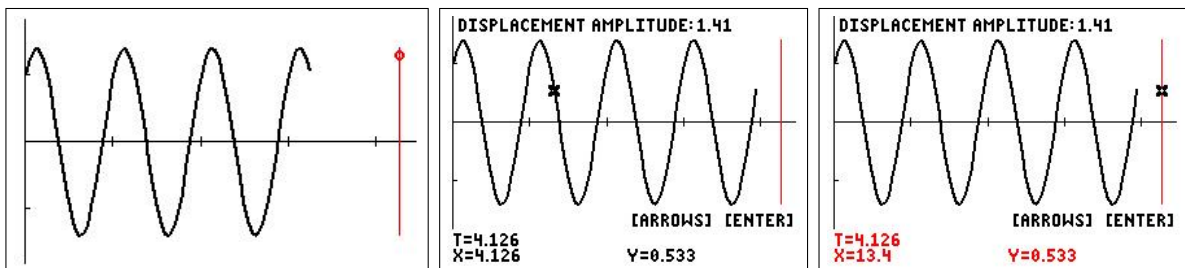
9.6.11   Bezier curves

The programs here should be used in conjunction with the article *Bezier curves: A classroom investigation* by Ed Staples in the Australian Senior Mathematics Journal 19(2), pages 59–64.[7] He used a feature of Excel at that time called spinners to generate the curves in the article and in a number of other applications.

The three programs here (CE only) allow two terminals (end points) and either one (SP-BZIER3), two (SPBZIER4) or three (SPBZIER5) control points. The three figures below reproduce the Bezier curves in Figures 3A, 5 and 6 of Ed's article, respectively.

---
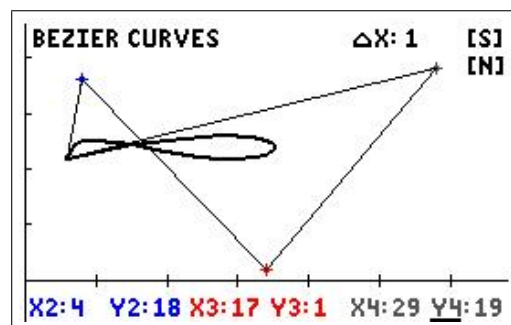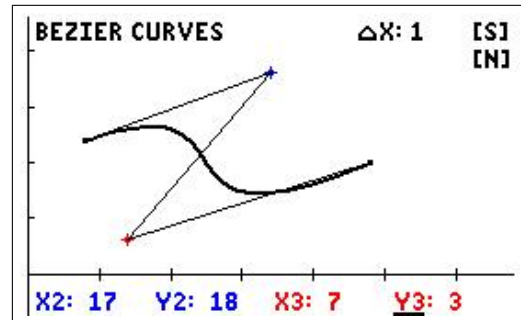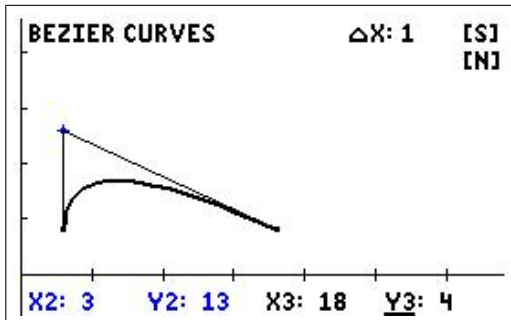
[7]available at *canberramaths.org.au* under *Resources*

9.7   Appendix 2: Graph details

Example 2   page 137
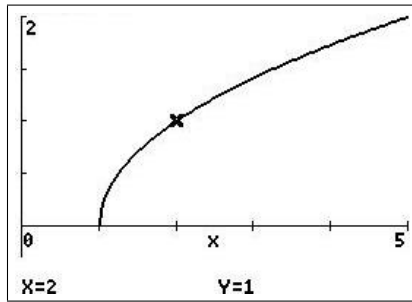

```
WINDOW
 Xmin=0
 Xmax=5
 Xscl=1
 Ymin=-0.3
 Ymax=2
 Yscl=0.5
```

■＼Y₁▉√(X−1)

Example 3   page 138

Y₁={(1−X);(X<1),(√(X−1));(X≥1)}

X=2   Y=1
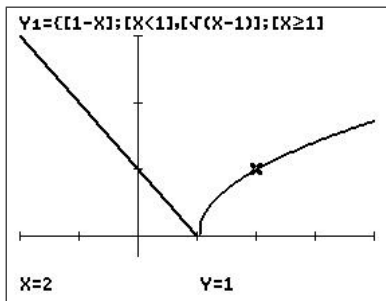
■＼Y₁▉piecewise(1−X,X<1,√(X−1),X≥1)

```
WINDOW
 Xmin=-2
 Xmax=4
 Xscl=1
 Ymin=-0.3
 Ymax=3
 Yscl=1
```

piecewise is ⌷math⌷ ⌷B⌷ on a CE; on an 84, $Y_1 = (1-X)(X<1)$, $Y_2 = \sqrt{(X-1)}(X≥1)$.

Leading-coefficient test for polynomial functions   page 9

UP TO THE LEFT   UP TO THE RIGHT
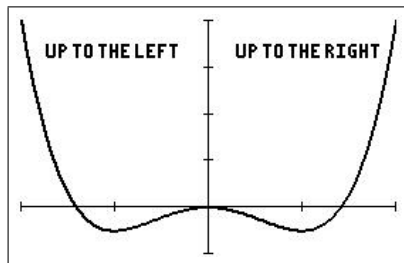
■＼Y₁▉4X^4−2X²

```
WINDOW
 Xmin=-1
 Xmax=1
 Xscl=0.5
 Ymin=-0.5
 Ymax=2
 Yscl=0.5
```


DOWN TO THE RIGHT
DOWN TO THE LEFT
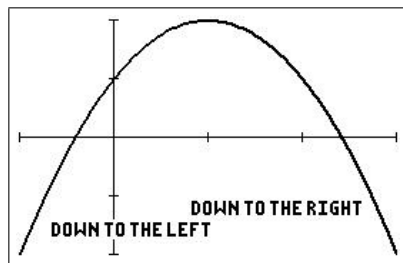
■＼Y₁▉2−(X−1)²

```
WINDOW
 Xmin=-1
 Xmax=3
 Xscl=1
 Ymin=-2
 Ymax=2
 Yscl=1
```

PTO

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UP TO THE RIGHT

DOWN TO THE LEFT
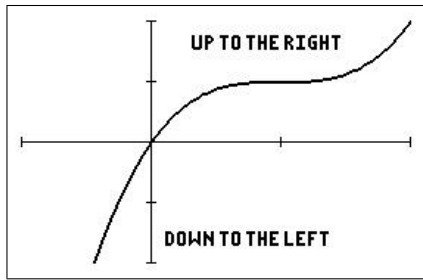
∎＼Y₁⊟(X−1)³+1

WINDOW
 Xmin=-1
 Xmax=2
 Xscl=1
 Ymin=-2
 Ymax=2
 Yscl=1

UP TO THE LEFT

DOWN TO THE RIGHT
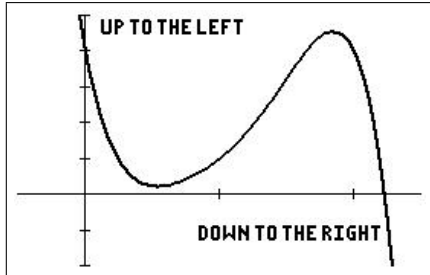
∎＼Y₁⊟-4(X−1)^5+X³

WINDOW
 Xmin=-0.5
 Xmax=2.5
 Xscl=1
 Ymin=-2
 Ymax=5
 Yscl=1

Zeros of a function   page 13

Uses *zero* in the $\boxed{\text{calc}}$ menu $\left(\text{on } \boxed{\text{trace}}\right)$.
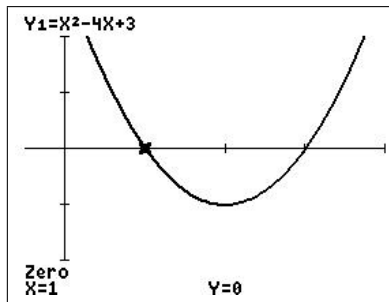
Y₁=X²−4X+3

Zero
X=1   Y=0

∎＼Y₁⊟X²−4X+3

WINDOW
 Xmin=-0.5
 Xmax=4
 Xscl=1
 Ymin=-2
 Ymax=2
 Yscl=1

CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx

Y₁=X²−4X+3

LeftBound?
X=0.8977273   Y=0.2150052

Y₁=X²−4X+3

Ri9htBound?
X=1.1363636   Y=-0.254132

Y₁=X²−4X+3

Guess?
X=1.0340909   Y=-0.06702

Y₁=X²−4X+3

Zero
X=1   Y=0

38
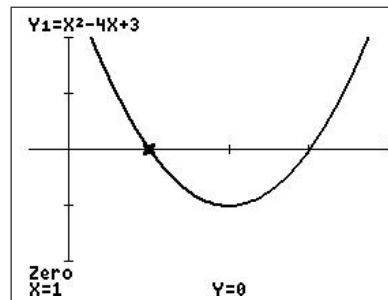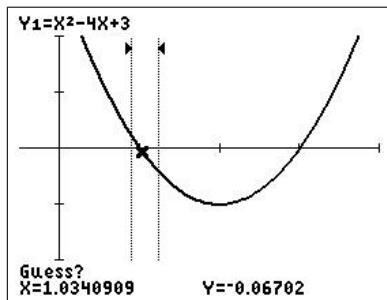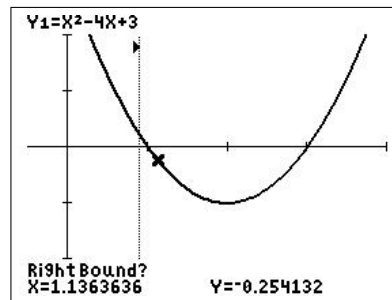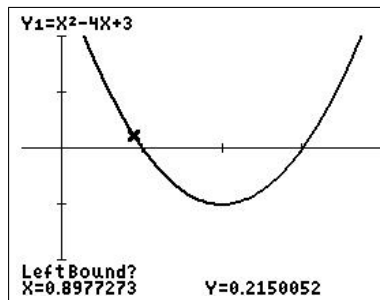
Greatest-integer function   page 15

int is in the math NUM menu.
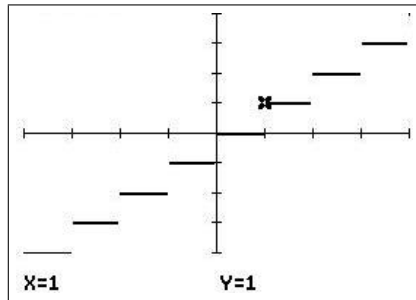
Least-integer function   page 15

There is no explicit least-integer function on the calculator, so I had to cheat a bit here. I plotted $Y = \text{int}(X) + 1$, which appears to give the graph of the least-integer function. However, if you trace to $X = 1$ on this graph, the function value is actually 2, not 1 as shown; 1 is the correct value. To get the correct value, I traced to $X = 0.9999999999$, which is displayed as 1 but has function value 1.
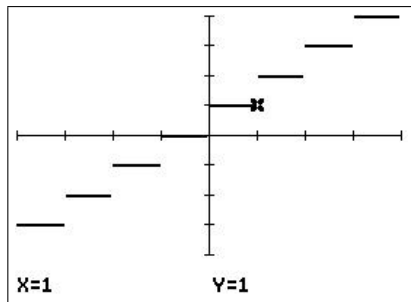
PTO

Examples 9 and 10   pages 16 and 17

Here I used the PERIODCE program.[8] It asks you to give the period, the phase shift (a period starts at $x = 0$ if the phase shift is 0) and the function to be made periodic (see the figures below). It then plots the periodic function over 4 periods. For the final graph, I then changed the window and pressed $\boxed{\text{graph}}$ to regraph the function (the function information is contained in $Y_1 - Y_3$).

```
 PERIODIC FUNCTION            PLOT FUNCTION          FUNCTION 0<X<P
1:PLOT FUNCTION                                     1:ALREADY IN Y1
2:TIDY UP                  PERIOD P: 1              2:ENTER NOW
                           PHASE SHIFT: 0
```



```
ENTER A FUNCTION

F(X)= X
```



```
WINDOW
 Xmin=-3
 Xmax=3
 Xscl=1
 Ymin=-0.12
 Ymax=1.2
 Yscl=0.5
```
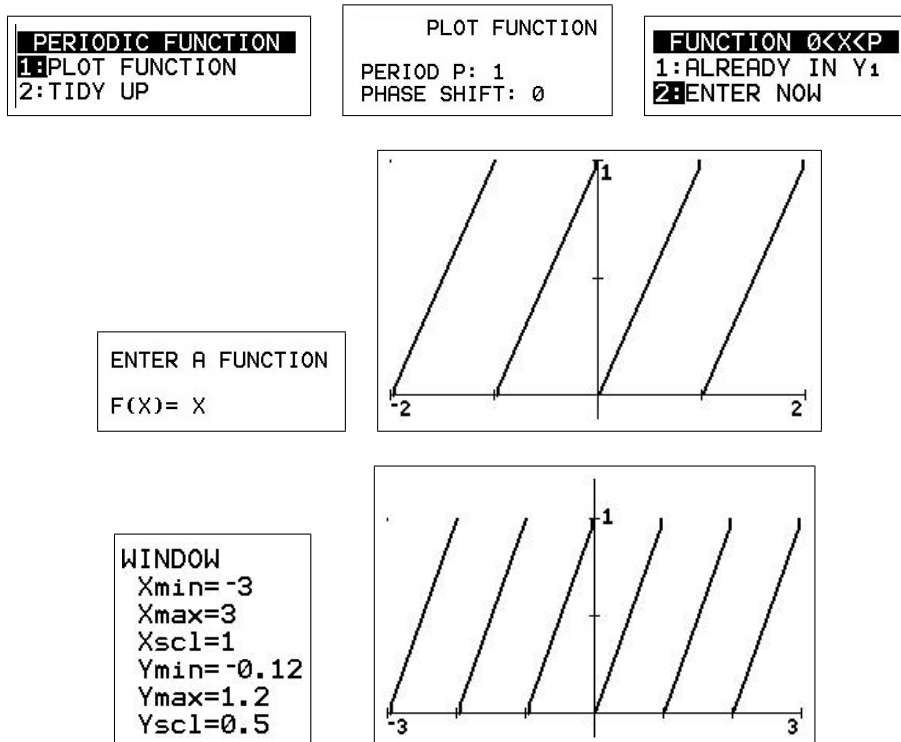
Heaviside function   page 40

The HEAVISDE/HEAVISCE program uses logical operators (see the calculator guidebook) to plot a specified function $f(X)$ that involves up to twenty Heaviside functions. The program takes as input a function f written in normal notation, with the Heaviside function written as H[X] — the brackets enclosing the argument of any Heaviside function must be square brackets.[9]

Use

Set a suitable window (may be changed later). Run the program and input the function $F(X)$ using square brackets for the arguments of the Heaviside function(s). Don't use square brackets for anything else. The independent variable must be X.

Press $\boxed{\text{enter}}$ to obtain a plot.

The Heaviside function is stored in $Y_1 - Y_3$ and can be replotted normally (using trace, changing windows, etc) after the program has finished running.

Technical Note: For ease of programming, in the plotted Heaviside functions $H(0) = 0$.

---

[8]PERIODIC is the corresponding program for the TI-84.

[9]The symbol you use for the Heaviside function can actually be anything you like — it is the square brackets that are used by the program.

Example 11

Page 104

1. $f(x) = H(x)$, the basic Heaviside function.
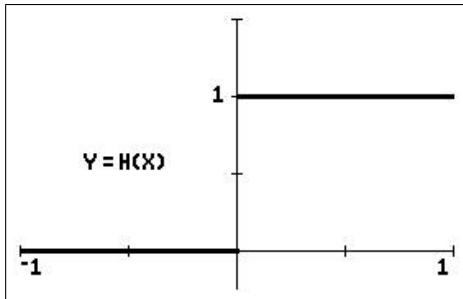
```
WINDOW
 Xmin=-1
 Xmax=1
 Xscl=0.5
 Ymin=-0.25
 Ymax=1.5
 Yscl=0.5
```

```
HEAVISIDE FUNCTION H[X]

F[X] CAN CONTAIN UP TO
20 HEAVISIDE FUNCTIONS

F[X] = H[X]
```
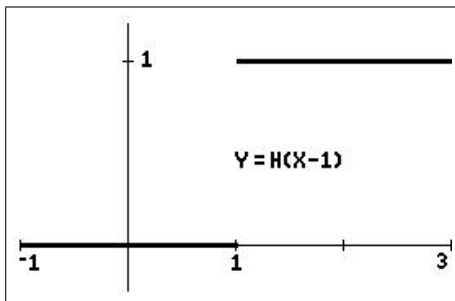
2. $f(x) = H(x-1)$.

```
WINDOW
 Xmin=-1
 Xmax=3
 Xscl=1
 Ymin=-0.25
 Ymax=1.2
 Yscl=1
```

```
HEAVISIDE FUNCTION H[X]

F[X] CAN CONTAIN UP TO
20 HEAVISIDE FUNCTIONS

F[X] = H[X-1]
```
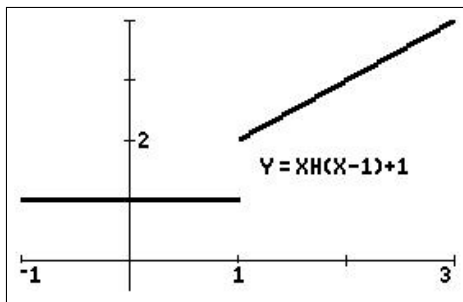
3. $f(x) = xH(x-1) + 1$.

```
WINDOW
 Xmin=-1
 Xmax=3
 Xscl=1
 Ymin=-0.5
 Ymax=4
 Yscl=1
```

```
HEAVISIDE FUNCTION H[X]

F[X] CAN CONTAIN UP TO
20 HEAVISIDE FUNCTIONS

F[X] = XH[X-1]+1
```
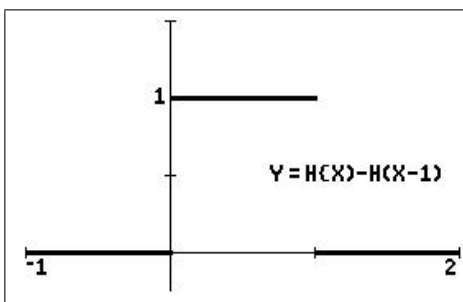
4. $f(x) = H(x) - H(x-1)$.

```
WINDOW
 Xmin=-1
 Xmax=2
 Xscl=1
 Ymin=-0.25
 Ymax=1.5
 Yscl=0.5
```

```
HEAVISIDE FUNCTION H[X]

F[X] CAN CONTAIN UP TO
20 HEAVISIDE FUNCTIONS

F[X] = H[X]-H[X-1]
```

Doing and saving the graphs

Almost all the figures were done in black and white on a TI-84CE. The CE has a broad border around the graphics part of the screen which can be one of four colours (⟨format⟩, BorderColor). I used Colour 4, which is white, so that the border was not evident.

I used TI Connect CE (available at *education.ti.com* under Downloads) to capture the screens. Saving the screen (see Section 16.2.1) produced a .png file. Open with *Paint*, click on the down arrowhead under *Image*, then the down arrowhead under *Select* and on *Rectangular selection*. With the cursor, starting at a corner of the part of the figure you wish to retain, pull out a box to cover the rest of it. Click on the *Crop* icon (top right-hand corner of the *Select* box) and save the resulting figure. You can also save it as a .jpg file or various other types.

10   Graph and Calculus Operations

10.1   Introduction

10.1.1   Australian Curriculum

References are given here to the corresponding topics in the Australian Curriculum. Specific references are given to the texts *Nelson Senior Maths Methods 11* (NSM11) and *Nelson Senior Maths Methods 12* (NSM12) used in the ACT.

The material here in Section 10.3 is generally relevant to the topics *Rates of change* (Chapter 10 in NSM11), *Properties of derivatives* (Chapter 10 in NSM11), *Applications of derivatives* (Chapter 12 in NSM11) and *Applications of derivatives* (Chapter 3 in NSM12).

10.1.2   Graphics calculators and Calculus

Graphics calculators lend themselves very nicely to demonstrating the important visual aspects of Calculus, graphs of functions, tangent lines, areas under curves, etc, without the off-putting effort required to do these by hand. They can also calculate numerically (approximately) many of the quantities that arise — derivatives and definite integrals, maximum and minimum values, etc.[10]

They can be used at a number of levels.

- As a sophisticated scientific calculator with many built-in functions.

- As a basic graph plotter — what does the graph of $y = e^x$ look like?

- As an advanced graph plotter able to plot Cartesian, parametric, polar and sequence graphs.

- To investigate 'what if' questions, for example what happens if you change the parameters a and b in the equation $y = ax^2 + b$?

- To do (numerically) many of the basic calculations in Calculus, such as finding the slope at a point on a graph, definite integrals, maxima and minima, solutions of equations, intersection points of graphs (simultaneous equations), etc.

- To illustrate graphically, perhaps by way of a program, some of the concepts of Calculus. Two examples are showing how a secant line tends to a tangent line in the appropriate limit and how we can approximate the area under a graph by the areas of some rectangles. With sufficient ingenuity, almost anything can be done here, the only limitation being the small screen of the calculator.

- To automate, using the built-in operations or programs, some of the calculations that arise in Calculus and other areas of Mathematics: numerical integration methods; fitting curves to data points; statistical functions for organising, analysing and displaying data; probability calculations; matrices; and so on.

---

[10]There are, of course, 'calculators' that go even further and do things symbolically — CAS calculators.

10.1.3   Calculator note

There are two options on the TI-84/CE that can help with the calculations here.

MATHPRINT

In CLASSIC mode, commands are typed on one line, with arguments in brackets. In MATH-PRINT mode, the calculator tries to display commands in mathematical notation, with small boxes in the relevant positions for the inputs. This applies to the powers and roots (blue) commands on the keyboard and to many of the commands in the $\boxed{\text{math}}$ NUM menu, and so is of some use in the operations here. Where it applies, it is illustrated with a screen shot.

Set CLASSIC or MATHPRINT in $\boxed{\text{mode}}$.

Catalog Help

An app on the TI-84Plus,[11] built in to the operating system of a TI-84CE, this tells you what arguments are required for most of the commands on the calculator in CLASSIC mode, allows you to input the arguments and paste the whole command onto the Home screen. Activated by scrolling down to the particular command and pressing $\boxed{+}$.

10.2   Graph operations

10.2.1   Graph $f(x) = \sin(2x)$ for $0 < x < \pi$

Note that the graphics and table keys are the top row of keys.

- Press $\boxed{\text{y=}}$: set $Y_1 = \sin(2X)$.

  The independent variable X is the $\boxed{X, T, \theta, n}$ key in the third row of keys.

  Note the highlighted = sign, which means the function will be plotted when you press $\boxed{\text{graph}}$.

  Move the cursor over the = sign and press $\boxed{\text{enter}}$ to toggle the function off/on.
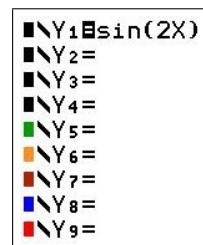
- Press $\boxed{\textbf{mode}}$ (*second row of keys*).

    - *Radian*: highlight with the cursor and press $\boxed{\text{enter}}$.

    - $\boxed{\text{2nd}}$ $\boxed{\text{quit}}$ to return to the Home screen.

---
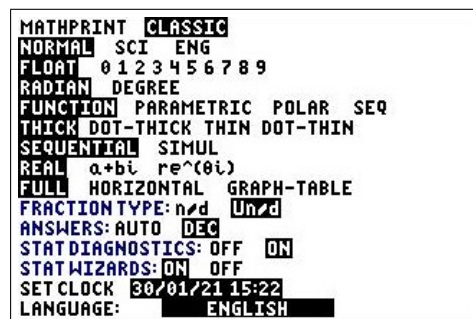
[11]download from *education.ti.com*

- Press window : specify the viewing window.

    – Suitable X values here are:
       Xmin $= 0$ Xmax $= \pi$ Xscl $= 0.5$.

       Use enter or the down arrow to move be-
       tween values. π is 2nd ∧ .

       Xscl is the distance between tick marks on the
       X axis (cosmetic only: 0 gives no tick marks).

    – Suitable Y values here are:
       Ymin $= -1.3$ Ymax $= 1.3$ Yscl $= 0.5$.

    – Note the difference between the blue subtract
       key − and the white change-sign key (–) .

    – The last three items in window are not rele-
       vant to everyday graphing.

```
WINDOW
 Xmin=0
 Xmax=3.141592654
 Xscl=0.5
 Ymin=-1.3
 Ymax=1.3
 Yscl=0.5
 Xres=1
 ▵X=0.011899972174242
 TraceStep=0.023799944348…
```

You could also use ZFit (zoom 0), which calculates appropriate Y values for the
function(s) and X values you set, or ZTrig (zoom 7), which sets axes suitable for
trigonometric functions.

- Press **graph** : graph the function.

  You can move the cursor around the screen with
  the left- and right-arrow keys; the cursor coordi-
  nates are shown at the bottom of the screen.

```
X=0.7853982        Y=0.4914634
```
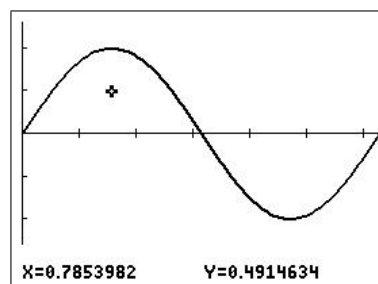
- Press **trace** : now the left- and right-
  arrow keys move the cursor along the
  graph; the coordinates of the point on
  the graph are shown at the bottom of the
  screen.

  If there is more than one graph, you move
  between them with the up- and down-
  arrow keys.

```
Y1=sin(2X)



X=0.7853982        Y=1
```

PTO

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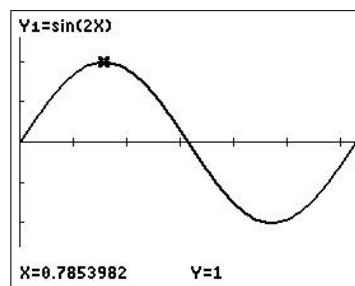
10.2.2 Estimate $f(1)$

- **On the graph**
  - Press $\boxed{\text{trace}}$.
  - Type in the X value, $\boxed{1}$ $\boxed{\text{enter}}$, to move to the desired point on the graph. Note the coordinates at the bottom of the screen.
  - Alternatively, use the left/right arrows to move the cursor along the curve (but note the problem when trying to reach X = 1).
  - The up/down arrows move between functions if there is more than one graphed.

```
Y₁=sin(2X)




X=1              Y=0.9092974
```
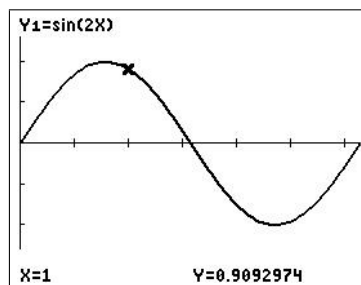
- **On the Home screen**
  - Type in Y₁(1) $\boxed{\text{enter}}$: Y₁ is in the $\boxed{\text{vars}}$ Y-VARS Function menu: $\boxed{\text{vars}}$ $\boxed{\rightarrow}$ $\boxed{1}$ $\boxed{1}$. You can't just type $\boxed{\text{Y}}$ $\boxed{1}$. $\boxed{\rightarrow}$ is the $\boxed{\text{sto}\rightarrow}$ key.

```
Y₁(1)
              0.9092974268
```

- **On a table**

  Set the table 'window' using $\boxed{\text{tblset}}$:
  TblStart = 0   ΔTbl = 0.5   Auto Auto.
  Select Auto with the cursor and $\boxed{\text{enter}}$.

```
TABLE SETUP
 TblStart=0
 ⧄Tbl=0.5
Indpnt: Auto Ask
Depend: Auto Ask
```

  Press $\boxed{\text{table}}$: scroll down in either column, up past the top in the X column.

  Move the cursor to the Y₁ column to see more digits in the Y value.

```
  X   | Y₁
  0   | 0
  0.5 | 0.8415
  1   | 0.9093
  1.5 | 0.1411
  2   | -0.757
  2.5 | -0.959
  3   | -0.279
  3.5 | 0.657
  4   | 0.9894
  4.5 | 0.4121
  5   | -0.544
Y₁=0.90929742682568
```

Answer: $f(1) = 0.90930$, to 5 decimal places.

Accuracy: 'at least five accurate digits' (TI-84 Guidebook).

PTO

10.2.3   Solve $\sin(2x) = 0.5$ for $0 \leqslant x \leqslant \pi$

- **On the graph**

    - Graph $Y_1 = \sin(2X)$ and $Y_2 = 0.5$.

    - Use *intersect* in the $\boxed{\text{calc}}$ menu: press $\boxed{\text{enter}}$ to select each curve; move the cursor to provide the *guess* or type in an X value and press $\boxed{\text{enter}}$. Your *guess* will determine which solution you find if there are several (as here).

      If you have more than two curves, select the ones you want with the up/down arrows before pressing $\boxed{\text{enter}}$.

- **From the Home screen**

    - Use the interactive *Numeric Solver*: $\boxed{\text{math}}$ $\boxed{0}$. You can also use *solve* in the $\boxed{\text{catalog}}$.

    - In CLASSIC mode, enter the equation as $0 = \sin(2x) - 0.5$ and press $\boxed{\text{enter}}$. Enter a guess for X (this will determine which of the two possible values you find) and, with the cursor on X, press $\boxed{\text{SOLVE}}$.

    - To find the second value, you have to provide a guess close enough to it (or change the bounds). The graphical method shows you where the intersection points are, and is therefore the preferred method.

    - *Answer*: the curves intersect at $x = 0.26180$ and $x = 1.30900$, both rounded to 5 decimal places. *Accuracy*: (claimed) 10^{-5}.
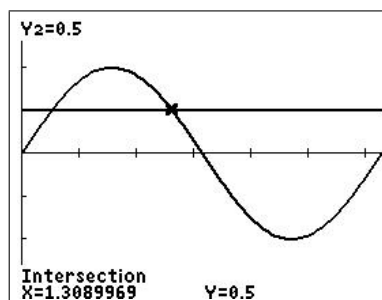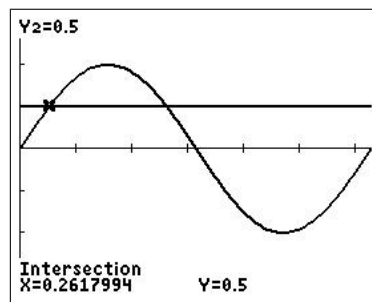
$\sin^{-1}(0.5)/2$ only gives you the first value; the second value is $\big(\pi - \sin^{-1}(0.5)\big)/2$.

Solver in MATHPRINT mode looks slightly different; it allows you to enter the two sides of an equation separately. After pressing $\boxed{\text{SOLVE}}$, the process is the same.

PTO

10.2.4   Find the first positive maximum of $f(x) = 3\sin(x^2)$

- **On the graph**
  - Set $Y_1 = 3\sin(X^2)$. Turn off or clear all other functions. Press $\boxed{\text{zoom}}$ $\boxed{7}$ (*ZTrig*) to obtain standard trig axes and plot the function.

  - Select *maximum* in the $\boxed{\text{calc}}$ menu. The calculator asks for a left bound, which you select by moving the cursor somewhere to the left of the maximum you want (or by typing in a value) and pressing $\boxed{\text{enter}}$. Similarly with the right bound and the guess. The guess doesn't have to be spot on. Use the bounds and the guess to pick out the (local) maximum you want.

  - Press $\boxed{\text{2nd}}$ $\boxed{\text{quit}}$ and type $2X^2$ $\boxed{\text{enter}}$ on the Home screen; X now contains the value found by *maximum*. You should recognise the first 6 digits of the number.


```
Y1=3sin(X²)



Maximum
X=1.2533155        Y=3
```
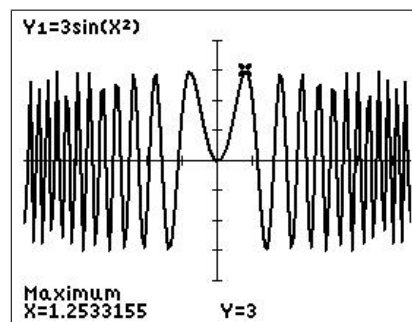
- **On the Home screen**
  - fMax(Y_1, X, 1, 1.5).

    fMax is in the $\boxed{\text{math}}$ menu; X is the variable you are maximising with respect to; and the last two inputs are the bounds for the search. These will determine which (local) maximum it will find; it won't tell you if there is more than one.

```
fMax(Y1,X,1,1.5)
            1.253314155
```

- *Answer*: the first positive maximum value of 3 occurs at $x = 1.25331$, rounded to 5 decimal places.

- *Accuracy*: (claimed) 10^{-5} — can be set in *fMax* by an optional fifth argument.

As with finding intersection points, the graphical method shows you which maximum you are finding, and again is the preferred method.

Note: The $\boxed{\text{calc}}$ operations *minimum* and *zero* work in the same way as *maximum*. *fMin* in $\boxed{\text{math}}$ menu works in the same way as *fMax*; these operations are the same in both CLASSIC and MATHPRINT modes.

10.3   Calculus operations

10.3.1   Estimate $f'(1)$

- **On the graph**

  - Set $Y_1 = \sin(2X)$ and $\boxed{\text{window}}$ $[0, \pi, 0.5] \times [-1.3, 1.3, 0.5]$.

  - Press $\boxed{\text{graph}}$.

  - Select dy/dx in the $\boxed{\text{calc}}$ menu $\left(\boxed{\text{2nd}}\ \boxed{\text{trace}}\right)$.

    Type in the X value: $\boxed{1}\ \boxed{\text{enter}}$.

    Alternatively, move the cursor to a point on the curve and press $\boxed{\text{enter}}$.

  - *Tangent* in the $\boxed{\text{draw}}$ menu draws the tangent at the point specified (as for dy/dx) and gives its equation. Adjust the number of decimal places displayed (second line in $\boxed{\text{mode}}$) to see the whole equation.

```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```

```
dy/dx=-0.832293
X=1          Y=0.9092974
```
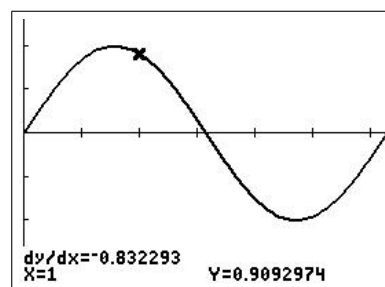
- **On the Home screen**

  CLASSIC mode

  - nDeriv $(Y_1, X, 1)$.

    nDeriv is in the $\boxed{\text{math}}$ menu: $\boxed{\text{math}}\ \boxed{8}$.
    The X is the variable we are differentiating Y_1 with respect to.

  - *Accuracy*: can be adjusted by an optional fourth argument h; the default value (top line in the figure) is 10^{-3}. In the second line, $h = 10^{-4}$.

  - You can use any independent variable here: e.g. the first derivative of $\sin(2t)$ at $t = 1$ is given by nDeriv$(\sin(2T), T, 1)$.

```
nDeriv(Y₁,X,1)
             -0.8322931182
nDeriv(Y₁,X,1,ᴇ⁻4)
             -0.8322936676
```

  MATHPRINT mode

  nDeriv gives the top screen, with boxes for the independent-variable name, the function and the value of the independent variable; inserting these and pressing $\boxed{\text{enter}}$ gives the bottom screen.

  In MATHPRINT, you can't adjust h; you have to accept that the answer you get is probably reasonably accurate.

```
d/d☐(☐)|☐=☐
```

```
d/dX(Y₁)|ₓ₌₁
             -0.8322931182
```

- *Answer*: $f'(1) = -0.83229$, to 5 decimal places.

nDeriv uses the symmetric difference quotient to approximate $f'(x)$; it is more accurate for a given h than the (forward) difference quotient used in the formal definition of the derivative.

10.3.2   Estimate $f''(1)$

On the Home screen

- The *nDeriv* command can be nested to give the second derivative (top screen)

  nDeriv(nDeriv $(Y_1, X, X), X, 1)$,

  or the MATHPRINT equivalent (bottom screen).

```
nDeriv(nDeriv(Y₁,X,X),X,1)
                   -3.637184855
⁻4sin(2)
                   -3.637189707
```

```
d/dx( d/dx(Y₁)|ₓ₌ₓ)|ₓ₌₁
                   -3.637184855
```

- With $Y_1 = \sin(2X)$, the command gives the second derivative of $\sin(2x)$ at $x = 1$ as -3.63718 (6 SD); the exact value is $-4\sin(2) = -3.63719$ (6 SD).

- *nDeriv* cannot be nested twice to give the third derivative; it would not be accurate anyway.

10.3.3   Graphing derivative functions

See Section 10.4.6.

10.3.4   Estimate $\displaystyle\int_0^\pi \sin(2x)\,dx$

- **On the graph**

  – Press graph .

  – Select $\int f(x)\,dx$ in the calc menu.

    Type in the lower and upper integration limits when prompted, followed each time by enter .


```
∫f(x)dx=⁻5.14E⁻14
[0,3.1415927]
```
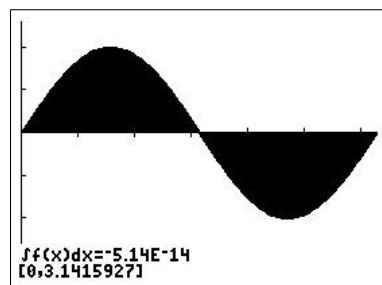
- **On the Home screen**

  – fnInt $(Y_1, X, 0, \pi)$ (CLASSIC; top screen).

    fnInt is in the math menu: math 9 .

    The X is the variable we are integrating Y_1 with respect to. In MATHPRINT, *fnInt* gives the template (middle screen) and, filled in, the result (bottom screen).

```
fnInt(Y₁,X,0,π)
                   -5.1379602ᴇ⁻14
```

```
∫▯▯(▯)d▯
```

```
∫₀^π(Y₁)dX
                   -5.1379602ᴇ⁻14
```

- *Answer*: $\displaystyle\int_0^\pi \sin(2x)\,dx = 0.00000$, rounded to 5 decimal places.

  The $\text{E}^{-}14$ stands for $\times 10^{-14}$.

- *Accuracy*: (claimed) 10^{-5} — can be adjusted in *fnInt* by an optional fifth argument (but not in MATHPRINT).

  More on definite integrals in *Numerical Integration*, Chapter 11.

10.3.5   Graphing definite integrals

See Section 10.4.6.

10.4   Activities

10.4.1   A classic problem

A hare and tortoise compete in a one-kilometre race. The distance each competitor has travelled from the starting point is given by a formula. In time t **minutes**, the distance in **metres** travelled by the hare is given by $H(t) = \frac{500}{3}(2\sqrt{t} + \sqrt[3]{t})$, while the distance in **metres** travelled by the tortoise is given by $T(t) = 100t + 250\sqrt{t}$.

Press $\boxed{\text{y=}}$ and enter the formulas for H and T. You have to use X $\left(\boxed{\text{X,T,}\theta\text{,n}}\right)$ as the independent variable. The cube root is $\boxed{\text{math}}$ $\boxed{4}$.

Set your $\boxed{\text{window}}$ so that the two graphs go from the bottom left to the top right of the screen. *Hints*: The race takes about 5 minutes. How far is the race?

If you select *Simul* in $\boxed{\text{mode}}$ before graphing, you will get a real-time view of the race. Choosing the graph style of a circle with a tail[12] for each competitor makes it even better.

Answer the following questions, writing down the steps you took. Plotting the lines $Y_3 = 500$, $Y_4 = 1000$ and using *intersect* in the $\boxed{\text{calc}}$ menu will be helpful. You may need to increase *Ymax* when using *intersect* so that the function formulas do not obscure the point you are interested in.

1. Who gets to the halfway point first? How long does it takes them? Verify your answer algebraically.

2. What is the time and distance at which the two runners are neck and neck?

3. Who wins the race, by what time margin and by what distance margin?

---

[12]Press $\boxed{\text{y=}}$, move the cursor to the left of Y1 and press $\boxed{\text{enter}}$ a couple of times on an 84; use the arrow keys on a CE. Repeat for Y2.

Teachers' notes

The questions in this version have been written in general terms deliberately for a good class. For a less-advanced class, students may need to be led a little through each question: *What equation do we need to solve to answer this question? What does this mean about the graphs of each side of the equation? How do we solve this equation on the calculator?* and so on.

Press $\boxed{\text{y=}}$ and put the equation for the hare in Y_1 and that for the tortoise in Y_2. Watch brackets here. You might like to discuss with the class how to write the formulas in a suitable form for the calculator. Time t becomes X on the calculator.

```
■\Y₁■500(2√(X)+³√(X))/3
■\Y₂■100X+250√(X)
```

Then set the $\boxed{\text{window}}$. Discuss first with the class what each axis represents and suitable scales. The Y axis is distance in metres, so $0<Y<1000$. The winner is then the competitor whose graph first reaches the top of the screen (providing *Simul* is set in $\boxed{\text{mode}}$).

```
WINDOW
 Xmin=0
 Xmax=5
 Xscl=1
 Ymin=0
 Ymax=1000
 Yscl=100
```

The time (X) scale has to be guessed. The race takes a little less than 5 minutes, so $0<X<5$ gives a good view. Set *Xscl*, the distance between the tick marks on the X axis to 1 and *Yscl* to 100. If either of these is too small, you will get a double line for the axis.

```
MATHPRINT CLASSIC
NORMAL  SCI   ENG
FLOAT  0 1 2 3 4 5 6 7 8 9
RADIAN  DEGREE
FUNCTION  PARAMETRIC  POLAR  SEQ
THICK  DOT-THICK  THIN  DOT-THIN
SEQUENTIAL  SIMUL
REAL  a+bi  re^(θi)
FULL  HORIZONTAL  GRAPH-TABLE
FRACTION TYPE: n/d  Un/d
ANSWERS: AUTO  DEC
STAT DIAGNOSTICS: OFF  ON
STAT WIZARDS: ON  OFF
SET CLOCK 30/01/21 16:31
LANGUAGE:        ENGLISH
```

1. From the graph (use $\boxed{\text{trace}}$ and the up/down arrows to see which graph is which), the hare clearly reaches the halfway point (500 m) first.


```
Y₁=500(2√(X)+³√(X))/3


          ×


X=0.9848485      Y=496.61907
```

To find how long the hare took, solve $H(t) = 500$ for t: set $Y_3 = 500$ and find the intersection of Y_1 and Y_3 using *intersect* in the $\boxed{\text{calc}}$ menu.
After you select *intersect*, the cursor will be on Y_1: press $\boxed{\text{enter}}$ to select it. The cursor will move to Y_2: press the down-arrow key to move it to Y_3 and press $\boxed{\text{enter}}$ to select it. Move the cursor to somewhere near the intersection and press $\boxed{\text{enter}}$ to provide a guess.

```
Y₃=500


          ×


Intersection
X=1              Y=500
```

The value for t is 1 minute, a value we can confirm algebraically to be exact by substituting $t=1$ into the equation for the hare. Note that it is easy to **verify** that $t=1$ is a solution, but tricky to **solve** $H(t)=500$ algebraically (it turns into a cubic equation).

The hare reaches the halfway point first in a time of 1 minute.

2. To find when they are neck and neck, we have to solve $H(t) = T(t)$, that is find the intersection of Y_1 and Y_2 (algebraically, this turns into a quartic equation). We obtain, using *intersect*, $t = 4.53$ minutes and distance equal to $985\,\mathrm{m}$ (both to 3 significant digits). It might be useful to increase *Ymax* or *Zoom In* ($\boxed{\text{zoom}}\ \boxed{2}$) on this part of the graph to see the two curves more clearly.
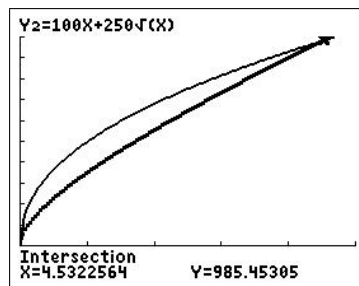
The hare and tortoise are neck and neck after about 4.53 minutes or about 4 minutes 32 seconds, at a distance of about 985 metres from the start.

3. To find the winner, we have to determine the time at which each competitor reaches the finish ($1000\,\mathrm{m}$).

Setting $Y_4 = 1000$, we find the hare finishes at $t = 4.681$ minutes (intersection of Y_1 and Y_4) and the tortoise finishes at $t = 4.624$ minutes (intersection of Y_2 and Y_4).
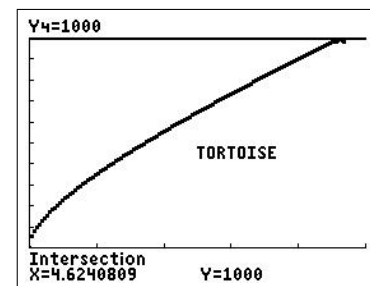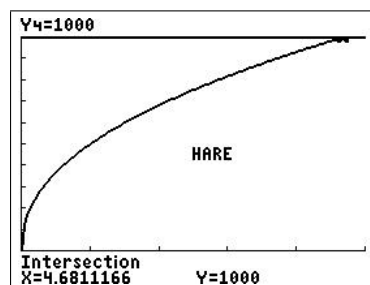
To find the distance margin, calculate $H(4.624)$, the position of the hare when the tortoise finishes: $H(4.624) \equiv Y_1(4.624) = 994.45\,\mathrm{m}$, rounded to 5 significant digits.

The tortoise wins the race by a margin of 0.057 minutes or 3.42 seconds. The distance margin is 5.55 m.

10.4.2   The best shape for a can

When manufacturers are designing their packaging, they must keep in mind the amount of product that has to fit inside and the amount of material it will take to make the package. Consider the humble soft-drink can. The standard volume is $375\,\text{mL}$ or $375\,\text{cm}^3$. Any number of cans can be designed that will hold this volume of liquid, but they will vary in shape and therefore in the amount of material needed to make the can (and therefore cost).

The formula for the volume of a cylinder V in terms of radius r and height h, is

$$V = \pi r^2 h.$$

Rearrange the volume formula to make h the subject and let the volume be $375\,\text{cm}^3$. *What are the units of r and h?*

$h =$

Enter this formula as Y_1, with X as radius r.
As a check, enter the volume formula: $Y_2 = \pi\,X^2\,Y_1$.
Y_1 is in the [vars] Y-VARS Function menu.

```
∎\Y₁⊟375⁄(πX²)
∎\Y₂⊟πX²Y₁
```

Press [tblset] ([2nd] [window]).
Set TblStart $= 1$ and ΔTbl $= 1$.
Press [table] ([2nd] [graph]).
Do you get the correct value for the volume in Y_2?

```
TABLE SETUP
 TblStart=1
 ⊿Tbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```

Write down the formula for the surface area of a cylinder, including the ends.*
The surface area determines the amount of material needed to make the can. *Why?* Enter the formula for surface area in Y_2 in terms of X (radius) and Y_1 (height).

SA $=$

$Y_2 =$

```
∎\Y₁⊟375⁄(πX²)
∎\Y₂⊟2πX(X+Y₁)
```

View the table of values again. *What do you notice about the values of the surface area?*

Use [tblset] to set new starting values and smaller increments to find the minimum surface area and corresponding radius (radius accurate to 1 decimal place).

Minimum SA $=$

Radius $=$

*To find the surface area of the side of the cylinder, think of making a vertical cut down the side from top to bottom, and spreading it out flat. What shape is it? What is its width? length?

Now graph the surface area as a function of radius, and use *minimum* in the [calc] menu to find the minimum. *Draw your graph below.*

Write down your values for the radius, height, surface area and circumference of the can when the surface area is a minimum. *Do these values seem reasonable?*

How does this compare with a can of soft drink? Why the differences? What about other cans?

You might like to read the article *The Best Shape for a Tin Can* by P. L. Roe, either in *The Mathematical Gazette* 75, 472 (1991) or reprinted in *The College Mathematics Journal* 24, 233 (1993). Search for it online.

Teachers' notes

This is an activity from *Integrating the Graphics Calculator into Year 9 and Year 10 of the Victorian Mathematics CSF*, Teachers Teaching with Technology (T^3), 1998, modified somewhat. There are a number of similar maximum/minimum activities for all years.

The height of the cylinder is given by $Y_1 = 375/(\pi X^2)$, where X is the radius r. The brackets are essential here because of the order in which the calculator does the different operations.

As a check, enter the volume formula $Y_2 = \pi X^2 Y_1$.

Press $\boxed{\text{table}}$.

X	Y₁	Y₂
1	119.37	375
2	29.842	375
3	13.263	375
4	7.4604	375
5	4.7746	375
6	3.3157	375
7	2.436	375
8	1.8651	375
9	1.4737	375
10	1.1937	375
11	0.9865	375

X=1

The total volume of the metal used to make the can, assuming the walls are of uniform thickness, is just the surface area times the thickness. Minimum surface area therefore means minimum volume of metal.

The surface area of a cylinder, including the ends, is given by $\text{SA} = 2\pi r^2 + 2\pi r h = 2\pi r(r+h)$.

```
∎\Y₁⊟375/(πX²)
∎\Y₂⊟2πX(X+Y₁)
```

X	Y₁	Y₂
1	119.37	756.28
2	29.842	400.13
3	13.263	306.55
4	7.4604	288.03
5	4.7746	307.08
6	3.3157	351.19
7	2.436	415.02
8	1.8651	495.87
9	1.4737	592.27
10	1.1937	703.32
11	0.9865	828.45

Y₂=288.03096491488

View the table of values again. What do you notice about the values of the surface area?

The surface area decreases then increases as the radius increases. There is a (local) minimum.

X	Y₁	Y₂
3	13.263	306.55
3.1	12.421	302.32
3.2	11.657	298.71
3.3	10.961	295.7
3.4	10.326	293.22
3.5	9.7442	291.25
3.6	9.2104	289.76
3.7	8.7192	288.72
3.8	8.2664	288.1
3.9	7.8479	287.87
4	7.4604	288.03

With TblStart $= 3$ and ΔTbl $= 0.1$, we find a radius of 3.9 cm for minimum surface area.

Alternatively, one can view the graph of surface area versus radius and use *minimum* in the $\boxed{\text{calc}}$ menu to find the minimum.

Y₂=2πX(X+Y₁)

Minimum
X=3.9079644   Y=287.87374
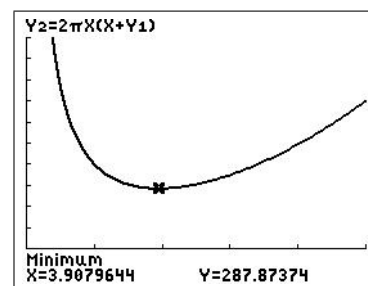
window $[0, 10, 2] \times [0, 1000, 100]$

Again, we obtain a value of $r = 3.9$ cm (accurate to 1 decimal place) for the radius giving minimum surface area.

If your students have sufficiently developed calculus skills, they could prove algebraically that the global minimum lies at $r = \sqrt[3]{375/2\pi} \approx 3.9$. More generally, for a given volume V, it is not too hard to show that $h = 2r$ (height = diameter) for the least surface area for a given volume.

Is this a reasonable answer for the radius?

Other factors may have to be taken into account such as what circumference is comfortable for the average human hand, the wastage of material when cutting the ends and the cost of making the joins. Some of these could be included in a more-complex model.

The values found here are:

radius $r = \sqrt[3]{\dfrac{375}{2\pi}} \approx 3.9\,\text{cm}$ height $h = \sqrt[3]{\dfrac{1500}{\pi}} \approx 7.8\,\text{cm}$ ratio $\dfrac{h}{r} = 2$

surface area $\approx 288\,\text{cm}^2$ circumference $\approx 24.6\,\text{cm}$

How do these compare with a can of soft drink?

A standard $375\,\text{mL}$ soft-drink can has a radius of $3.25\,\text{cm}$ and a height of $13\,\text{cm}$: $h/r = 4$. Its surface area is about $332\,\text{cm}^2$ and circumference $20.4\,\text{cm}$.

The article *The Best Shape for a Tin Can* by P. L. Roe, either in *The Mathematical Gazette* 75, 472 (1991) or reprinted in *The College Mathematics Journal* 24, 233 (1993)[13] goes into why there might be differences between the theory here and the actual values. A good example of mathematical modelling.

---

[13]search for it online

10.4.3 Slope of a graph

The program DIFFQPLT/DQPLTCE illustrates graphically how the difference quotient (the slope of the secant line, a line between two points) approaches the derivative (the slope of the tangent line) at a point on a graph.

- Put the function $f(x) = e^x$ in Y1. We wish to find the slope of the graph of f at $x = 1$.

  ▪\Y1◻e^(X)

- Run the program — press [prgm], then the number against DIFFQPLT/DQPLTCE or scroll down and press [enter].

  Press [enter] to start the program.

  DIFFERENCE QUOTIENT OF Y1
  BETWEEN X AND X+◦X

  X:

- Set $X = 1$ and $\Delta X = 2$. The program will plot the function for $1 < X < 1 + \Delta X$.

  ◦X=2
  SECANT SLOPE
  IS 8.6836275

  X+◦X

- Press [enter] to see the secant line and its slope.

- Keep pressing [enter] to repeat the process, but with ΔX reduced each time by a factor of 10. The box in the lower left-hand corner shows the area that will be graphed at the next step.

  ◦X=0.2
  SECANT SLOPE
  IS 3.0091755

  X+◦X

- When you've finished, select QUIT from the menu.

  NEXT...
  1:ZOOM IN
  2:QUIT

As we Zoom In more and more, the graph of f becomes straighter and straighter, so that it looks more and more like its tangent. The secant line, for which we can calculate the slope, approaches the tangent, so that the slope of the secant line is ultimately a good approximation to the slope of the tangent.

Try the program with other functions in Y1.

A convincing demonstration that even a nasty curve becomes straight if you magnify it enough (local linearity), provided the function is differentiable

Plot the graph of sin(20X) (Radian mode), using a window of $[0, 5, 1] \times [-1, 1.5, 0.5]$. Put the cursor on one of the peaks and successively Zoom In at the maximum. The sharp peak eventually looks like a (horizontal) straight line (the tangent to the curve). The function is differentiable there.

A contrasting function on which to try the same process is $y = 1 - (x - 1)^{2/3}$. Zoom In successively at the peak. *Is this function differentiable at $x = 1$?*

What about $y = |x|$ (abs(X)) at $x = 0$?

10.4.4   Discovering the derivative by exploration

Modified from: Judy Broadwin, *Discovering the derivative by exploration*, TI-82/83 Activities for Calculus.

We put several different functions $f(x)$ in Y1 and try to discover their derivatives using the difference quotient
$$\mathrm{DQ} \;=\; \frac{f(x+h) - f(x)}{h},$$
where h is a small number. We know that the derivative is the limit of DQ as $h \to 0$.

- Store 0.0001 in memory H: 0.0001 sto alpha H enter .

- Enter the difference quotient into Y2: $\mathrm{Y2} = \big(\mathrm{Y1(X+H)} - \mathrm{Y1(X)}\big)/\mathrm{H}$. Watch brackets! Y1 is in the vars Y-VARS Function menu. You can't just type the letter Y .

- Turn off Y2: move the cursor over the = sign and press enter .

Exploration 1   *discover the derivative of* $f(x) = e^x$

Set Y1 = $e^\wedge$(X).
Set window to $[-1, 2, 1] \times [-1, 5, 1]$.
Press trace .
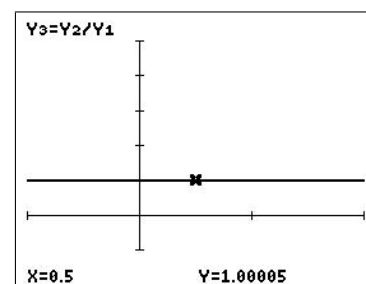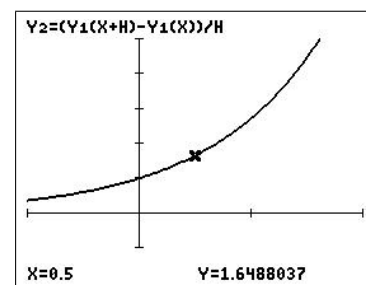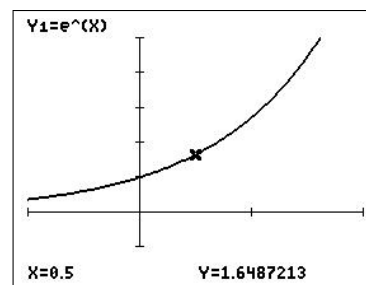
Then turn off Y1 and graph Y2. *What do you see?*

Turn off Y2, set Y3 = Y2/Y1 and graph it.
What do you see?

What is $\dfrac{d}{dx}\, e^x$ *?*

Exploration 2   *discover the derivative of $f(x) = \sin(x)$*

Set $Y_1 = \sin(X)$.
Turn Y_1 off and Y_2 on.

```
■\Y1=sin(X)
■\Y2目(Y1(X+H)-Y1(X))/H
■\Y3=cos(X)
```

```
Y2=(Y1(X+H)-Y1(X))/H
```

Press $\boxed{\text{zoom}}$ $\boxed{7}$, then $\boxed{\text{trace}}$.
Describe the graph of Y_2.
Do you recognise the function?

```
X=0                    Y=1
```
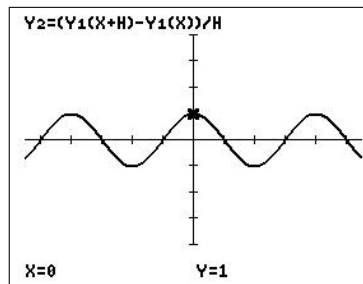
Can you describe the relationship between the graph of the function and the graph of its difference quotient?

X	Y2	Y3
0	1	1
1	0.5403	0.5403
2	-0.416	-0.416
3	-0.99	-0.99
4	-0.654	-0.654
5	0.2837	0.2837
6	0.9602	0.9602
7	0.7539	0.7539
8	-0.146	-0.146
9	-0.911	-0.911
10	-0.839	-0.839

```
Y2=-0.9899995509
```

Set $Y_3 = \cos(X)$.
Compare Y_2 and Y_3 using $\boxed{\text{table}}$.
Why are they not identical?

Exploration 3   *discover the derivative of $f(x) = \ln(x)$*

Set $Y_1 = \ln(X)$ and turn it off. Leave Y_2 on.

X	Y2
0	ERROR
1	1
2	0.5
3	0.3333
4	0.25
5	0.2
6	0.1667
7	0.1429
8	0.125
9	0.1111
10	0.1

```
X=0
```

In $\boxed{\text{tblset}}$, set TblStart $= 0$, ΔTbl $= 1$.
Press $\boxed{\text{table}}$.
Can you guess the derivative of $\ln(x)$?

10.4.5   Derivatives and the shape of a graph

From: *This is I.T. Graphics-calculator activities for upper-secondary students* by Pat Forster, Alan Cadby and Gary Young, AAMT, 1998.

This investigation is to help you understand how the first and second derivatives of a function can tell you the shape of its graph. You will also learn some new terminology to describe graphs.

1. Copy from your calculator the graph of $y = 5x^2 - 2x^5$ for $-1 \leqslant x \leqslant 2$, $-10 \leqslant y \leqslant 10$. Put scales on your graph.
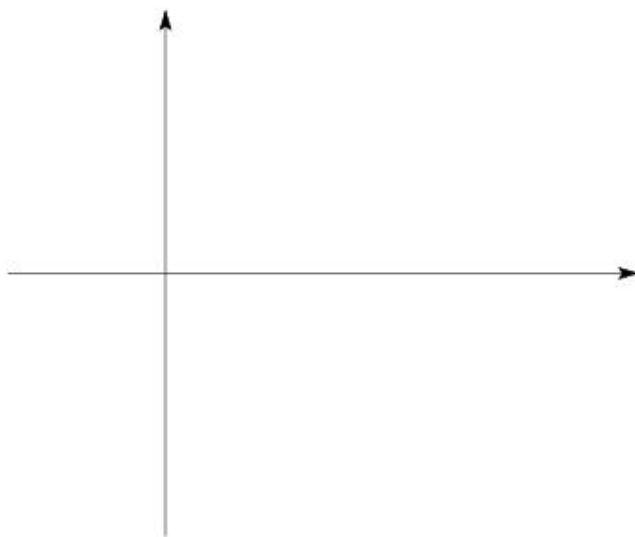
2. Find and enter the equations for the first and second derivatives of $y = 5x^2 - 2x^5$ into your calculator as Y2 and Y3. Look at the values of these derivatives for different values of x using the calculator's table of values as follows:

   First press $\boxed{\text{tblset}}$ and set TblStart $= -1$; ΔTbl $= 0.2$; then press $\boxed{\text{table}}$.

   (a) Use a red pen and put plus signs $(+)$ along the sections of your graph above where dy/dx is positive. Similarly, put minus signs $(-)$ and zero (0), where appropriate.

   (b) Using a different-coloured pen, mark where d^2y/dx^2 is $+$, $-$ or 0 (scroll up the table in the X column).

   (c) The points on a graph where $dy/dx = 0$ are called **stationary points**. Fill in the table below.

Type of stationary point	Co-ordinates	$\dfrac{dy}{dx} \ (+, -, 0)$	$\dfrac{d^2y}{dx^2} \ (+, -, 0)$
Maximum			
Minimum			

3. (a) Sections of a graph where $\dfrac{d^2y}{dx^2}$ is positive are described as **concave up**;

if $\dfrac{d^2y}{dx^2}$ is negative, the curve is **concave down**.

For what values of x is your graph concave up?

For what values of x is your graph concave down?

Is the graph at the maximum point concave up or concave down?

Is the graph at the minimum point concave up or concave down?

(b) A point at which a curve is concave down (d^2y/dx^2 negative) on one side and concave up (d^2y/dx^2 positive) on the other side, *or vice versa*, is called a **point of inflection**. The sign of the second derivative must *change* as we pass through a point of inflection because the graph changes concavity.

Use the table of values to find the point of inflection of the function here: change *TblStart* and ΔTbl to zoom in on the zero of Y₃. *Why do we look for a zero? Is this zero a point of inflection?* Write its co-ordinates on your graph.

(c) *Is $x=0$ a point of inflection of the function $y=x^4$? Give reasons.*

(d) Points of inflection can be the steepest section of a graph — look at the graph of $y=\sqrt[3]{x-1}$ at $x=1$ (the cube root is $\boxed{\text{math}}\ \boxed{4}$). They can also be stationary points (graph is horizontal) — look at the graph of $y=x^3+1$ at $x=0$. However, in general, the slope at a point of inflection can be any value.

Put a cross on the points of inflection in each of the following two graphs. *Is the point of inflection in either graph a stationary point?*
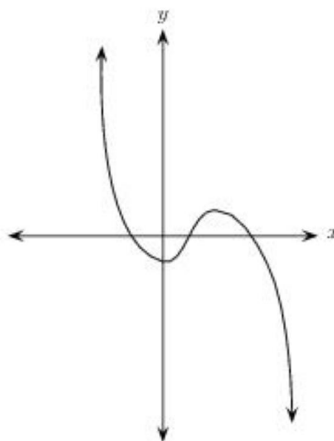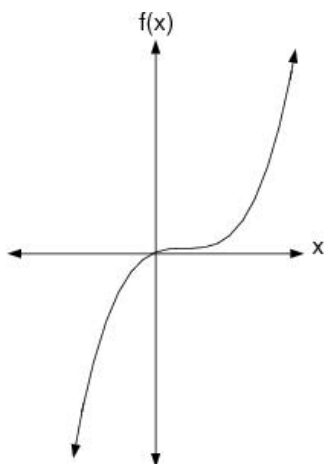
4. Sketch a graph with the given properties.
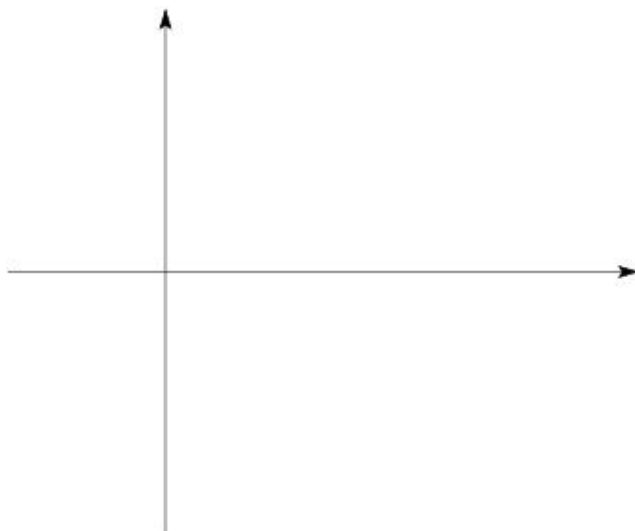
(a) Endpoints $(-2, 1)$ and (7,6).

(b) Stationary point at $(1, -5)$ where $\dfrac{d^2y}{dx^2}$ is positive.

(c) Stationary point at (3,2) where $\dfrac{d^2y}{dx^2} = 0$.

(d) $\dfrac{d^2y}{dx^2} = 0$ at $(2, -1)$, negative between $x=2$ and $x=3$, and positive for $x>3$.

(e) Classify the points in (b) – (d) as a maximum, minimum and/or point of inflection.

(f) For what values of x is the curve concave up?

10.4.6 Graphing derivatives and anti-derivatives

Based on: *How do you graph derivatives and anti-derivatives?* by John Maloney, Eightysomething! 7(1), 1997.

Here are two questions to challenge your students' understanding of the concepts of derivative and anti-derivative.

Question 1: The figure shows graphs of a function, its derivative and its second derivative.
Which curve is which?
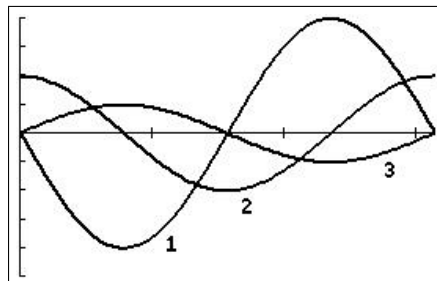
Question 2: The figure shows graphs of a function, its derivative and an anti-derivative.
Which curve is which?
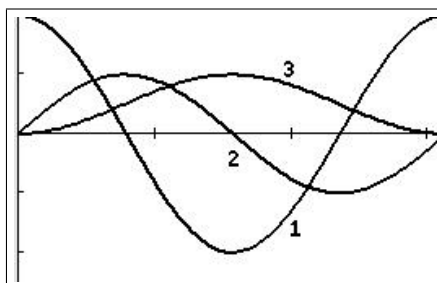
Let's investigate how to graph a function, its derivative and anti-derivative. Press $\boxed{\text{mode}}$ and make sure that *Radian* and *Func*(*tion*) are selected. Choose whichever of MATHPRINT or CLASSIC you prefer.

Press $\boxed{\text{y=}}$: set $Y_1 = \sin(2X)$ and a window of $[0, \pi, 1] \times [-5, 4, 1]$. Press $\boxed{\text{trace}}$.
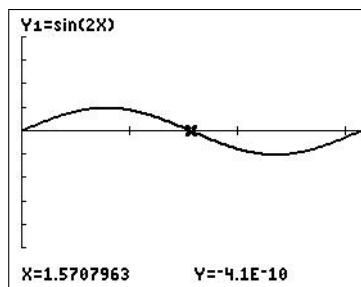
Plotting derivatives

To graph the first derivative of Y_1, set
$Y_2 = \text{nDeriv}(Y_1, X, X)$ or the MATHPRINT
equivalent (Section 10.3.1).

nDeriv is $\boxed{\text{math}}$ $\boxed{8}$. Y_1 is the function we are differentiating, the first X means differentiate with respect to variable X and the second X is the value to calculate the derivative at: this value is set by the grapher as it plots points on the graph.

Pressing $\boxed{\text{trace}}$ gives you the figure.
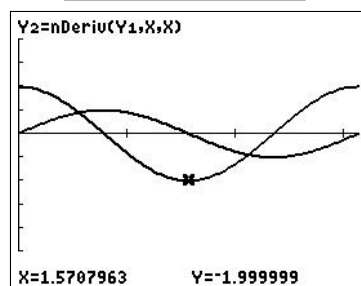
Set $Y_3 = \text{nDeriv}\,(Y_2, X, X)$ (or the MATHPRINT
equivalent) for the second derivative of Y_1 — the
derivative of the first derivative Y_2 (figures below).
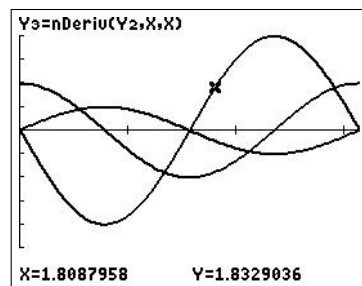
Note: The TI-84/CE will not allow you to calculate the third derivative in this manner —
it would not be very accurate anyway. You need to use a different difference approximation[14]
to the central or symmetric difference approximation used in *nDeriv*.

With Y_1, Y_2 and Y_3 turned on, as in the figures below, you should obtain the graph of the
first question above. Students, of course, must use the relationship between a function and
its derivative to answer the question.

Plotting an anti-derivative

To plot an anti-derivative of Y_1, set $Y_4 = \text{fnInt}\,(Y_1(T), T, 0, X)$ or the MATHPRINT equivalent
(Y_4 in the figures above). *fnInt* is $\boxed{\text{math}}$ $\boxed{9}$. Y_1 is the function we are integrating,[15] T a
dummy integration variable, and 0 and X are the integration limits.

The function we are plotting is therefore $y(x) = \displaystyle\int_0^x \sin(2t)\,dt$.

Turn off Y_2 and Y_3, set $\text{Ymin} = -2.5$, $\text{Ymax} = 2$ and
press $\boxed{\text{graph}}$. Setting $\text{Xres} = 3$ (plotting every third
point) in $\boxed{\text{window}}$ speeds up this graph without signifi-
cant loss of resolution. $\boxed{\text{on}}$ stops graphing.
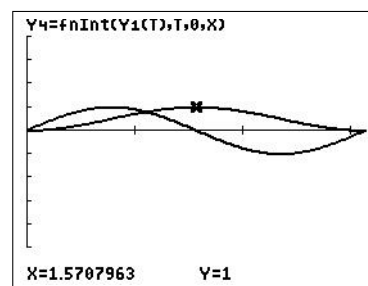
Note: The lower integration limit 0 is arbitrary. Change it to other numbers to show students
that a function has (infinitely) many (related) anti-derivatives. The one graphed here is the
one that passes through the origin.

With Y_1, Y_2 and Y_4 turned on, you should obtain the graph of the second question above.

Note: fnInt allows any letter to be used as the (dummy) integration variable; we can therefore
code the integral command the same as we write the integral: dummy variable T, independent
variable X. The integration routine provides values for T, the grapher values for X.

---

[14] $f'''(x) \approx \big(f(x+2\Delta x) - 2f(x+\Delta x) + 2f(x-\Delta x) - f(x-2\Delta x)\big)/2(\Delta x)^3$.

[15] We could have put the function in directly as sin(2T). Similarly with the function in *nDeriv*. The advantage
of the way we have set up the functions $Y_2 - Y_4$ is that we can see the derivatives and anti-derivative of whatever
function is in Y_1.

The Chain Rule

With $Y_1 = \sin(2X)$, set

$\qquad Y_5 = X^2$

$\qquad Y_6 = Y_1(Y_5)$

$\qquad Y_7 = \mathrm{nDeriv}(Y_1, X, Y_5) * \mathrm{nDeriv}(Y_5, X, X)$

$\qquad Y_8 = \mathrm{nDeriv}(Y_6, X, X).$

Plot and compare the graphs of Y_7 and Y_8 using a window of $[-3, 3, 1] \times [-12, 12, 4]$. Use $\boxed{\text{trace}}$ and the up/down arrows to identify the two curves. Explain what you see.

Answers to the questions

Question 1   *One possible answer*

Curve 3 is initially positive, so that it cannot be the derivative of either Curve 1 or Curve 2, both of which have negative slopes initially. Therefore, Curve 3 must be the function.

The initial slope of Curve 3 is positive, so that Curve 1, which is initially negative, cannot be the first derivative of Curve 3. Therefore, Curve 2, which is initially positive, must be the first derivative of Curve 3.

Curve 1, initially negative, must be the derivative of Curve 2, and therefore the second derivative of Curve 3.

Question 2   *One possible answer*

Curve 3 cannot be the derivative of either Curve 1 or Curve 2 because it does not pass through zero when either of these curves has a maximum/minimum. Therefore, Curve 3 must be either the function or an anti-derivative.

If Curve 3 were the function, Curve 2 could be its derivative (although its behaviour at either end doesn't look correct), but Curve 1 could not be an anti-derivative because it is decreasing initially, whereas the anti-derivative of (positive) Curve 3 must be increasing. Therefore, Curve 2 must be the function and Curve 3 an anti-derivative.

The anti-derivative of Curve 2 must increase where Curve 2 is positive and decrease where Curve 2 is negative. Curve 3 has these properties. Because the area of Curve 2 below the x axis is equal to the area above the x axis, the final y value of Curve 3 must be equal to its initial y value.

Curve 1 must therefore be the derivative of Curve 2. *Check*: when Curve 2 has a maximum or minimum, Curve 1 is zero; where Curve 2 is increasing, Curve 1 is positive; where Curve 2 is decreasing, Curve 1 is negative.

11   Numerical Integration

11.1   Introduction

Finding the area under a curve was one of the challenges and triumphs of Calculus. Starting with rectangles under and touching the curve, the sum of the areas of these rectangles, known as a Riemann sum, approximates the area under the curve. The limit of the Riemann sums as the number of rectangles tends to infinity and the width of the rectangles tends to zero is known as the definite integral. The Fundamental Theorem of Calculus allows this definite integral to be found exactly in terms of the anti-derivative of the function being integrated.

However, in some cases, the anti-derivative cannot be found or is very complicated. Going back one step to the rectangles, and calculating the sum of their areas with more and more, thinner and thinner rectangles gives us a simple approximation to the integral and some idea of the error in the approximation; this is known as the Left-Endpoint Rule if the top left-hand corner of each rectangle touches the curve, the Right-Endpoint Rule if the top right-hand corner of each rectangle touches the curve and the Midpoint Rule if the middle of the top of each rectangle lies on the curve.

One finds, however, that many rectangles are needed to find the integral to any degree of accuracy in most cases, and looks to methods that converge faster. Rather than approximating the function on each sub-interval (rectangle base) by a horizontal straight line (the top of the rectangle), a polynomial of degree 0, we can try approximating the function by a general straight line, a polynomial of degree 1, giving the Trapezoidal Rule, or by a parabola approximating the function over two sub-intervals, a polynomial of degree 2, giving Simpson's Rule. Simpson's Rule is the workhorse of numerical integration in many cases.

To improve on Simpson's Rule, we move away from a geometric approach; Numerical Analysis gives us methods such as Gauss quadrature, which are significantly more accurate than Simpson's Rule for the same number of function evaluations but still calculate a weighted sum of function values.

All these methods are easy to implement on a graphics calculator; a simple program in all cases allows us to vary easily the number of sub-intervals/function evaluations so that the error can be quantified. That is the subject here. Without a program, the hand calculations for even the simplest method are too lengthy to allow any sort of meaningful use.

11.1.1   Australian Curriculum

The material here is relevant to the topic *Integration and areas*, Chapter 4 in *Nelson Senior Maths Methods 12*, used in the ACT.

11.2   Methods and programs

The numerical-integration methods here all calculate approximations to the definite integral

$$\int_a^b f(x)\,dx. \tag{1}$$

11.2.1   Geometric methods

These methods rely on the geometric fact that the definite integral, Eq. (1), is equal to the (signed) area between the graph of the function f and the x axis, between $x=a$ and $x=b$.

The integration interval $[a, b]$ is partitioned (divided) by $N+1$ equally spaced points into N sub-intervals each of length h, where $h=(b-a)/N$. The function is then approximated by a polynomial of order 0 or 1 on each sub-interval or order 2 on each two sub-intervals.

1. Left- and Right-Endpoint Rules

Here rectangles, with bases the sub-intervals of width h, are drawn with the top left-hand corner of each rectangle touching the graph of f (Left-Endpoint Rule) or the top right-hand corner of each rectangle touching the graph of f (Right-Endpoint Rule), as shown in the figures below with 4 rectangles/sub-divisions. The integral is approximated by the sum of the (signed) areas of the N rectangles.
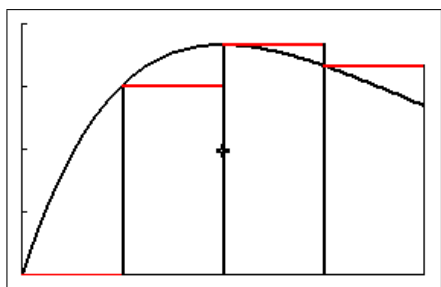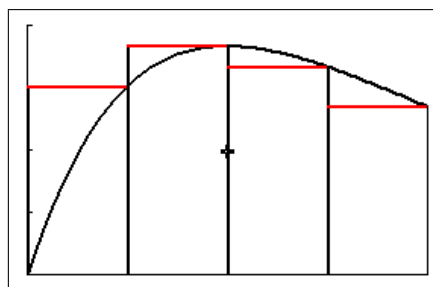
Left-Endpoint Rule   Right-Endpoint Rule

Another way of looking at this is that the function is approximated on each sub-interval by a horizontal straight line (red lines in the figures), a polynomial of degree 0. Clearly, the larger the number of rectangles N (and the smaller h), the better the approximation to the function on each sub-interval and the better the approximation the sum of the areas is to the integral.

This is the basic method, and gives rise to Riemann sums (the sum of the (signed) rectangle areas) and the definition of the definite integral as the limit of the sum of the areas of the rectangles as $N \to \infty$ and $h \to 0$.

Program: NINTGRPH/NINTGRCE draws the rectangles (Left- or Right-Endpoint Rule) and calculates the sum of their (signed) areas as an approximation to the integral of the function in Y1 over the given integration range using the Left- and Right-Endpoint Rules with N rectangles/sub-intervals. You specify N.

Program: NUMINT/NUMINTCE calculates and displays (without graphics) the approximations to the integral of Y1 over the given integration range using the Left- and Right-Endpoint Rules with N (specified) rectangles/sub-intervals.

Activity 1 (Section 11.4.1) goes into more detail on the Left- and Right-Endpoint Rules.

2. Midpoint Rule

The same as the Left- and Right-Endpoint Rules but the midpoint of the top of each rectangle lies on the the graph of f, as shown in the figure below with 4 rectangles/sub-divisions. Again, the integral is approximated by the sum of the (signed) areas of the N rectangles. In general, this is more accurate than the Left- and Right-Endpoint Rules for a given N.
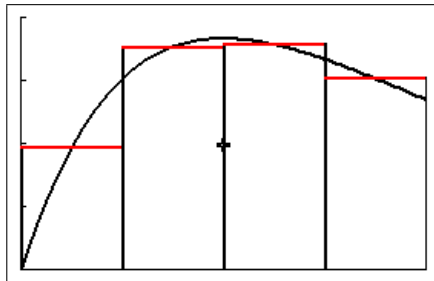
Midpoint Rule

Here, the function is also approximated on each sub-interval by a horizontal straight line, a polynomial of degree 0.

Program: NINTGRPH/NINTGRCE draws the rectangles and calculates the sum of their (signed) areas as an approximation to the integral of the function in Y_1 over the given integration range using the Midpoint Rule with N rectangles/sub-intervals. You specify N.

Program: NUMINT/NUMINTCE calculates and displays (without graphics) the approximation to the integral of Y_1 over a specified integration range using the Midpoint Rule wirh N (specified) rectangles/sub-intervals.

3. Trapezoidal Rule

Here, a straight line joins the points on the graph of f corresponding to each endpoint of a sub-interval (red lines in the figure below), giving a trapezium, as shown in the figure with 4 trapeziums/sub-divisions. The integral is approximated by the sum of the (signed) areas of the N trapeziums. The Trapezoidal Rule is the mean of the Left- and Right-Endpoint Rules.
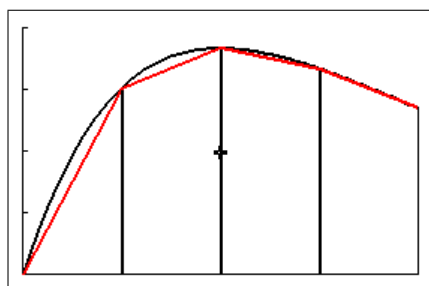
Trapezoidal Rule

Here, the function is approximated on each sub-interval by a straight line, a polynomial of degree 1.

Program: NINTGRPH/NINTGRCE draws the trapeziums and calculates the sum of their (signed) areas as an approximation to the integral of the function in Y_1 over the given integration range using the Trapezoidal Rule with N (specified) trapeziums/sub-intervals.

Program: NUMINT/NUMINTCE calculates and displays (without graphics) the approximation to the integral of Y_1 over a specified integration range using the Trapezoidal Rule with N (specified) trapeziums/sub-intervals.

4. Simpson's Rule

Here, a parabola or quadratic function, a polynomial of degree 2, is fitted to each set of 3 points on the graph of the function corresponding to the endpoints of two adjacent sub-intervals. The integral is then approximated by the sum of the (signed) areas between the $N/2$ parabolas and the x axis. The number N of sub-intervals must therefore be even. Simpson's Rule is, in general, more accurate than the methods above for a given (even) number of sub-intervals.
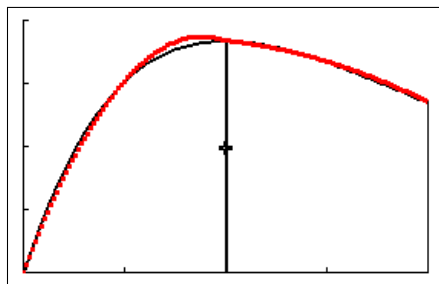
Simpson's Rule showing two parabolas (red) over 4 sub-intervals

Program: NINTGRPH/NINTGRCE draws the parabolas and calculates the sum of their (signed) areas as an approximation to the integral of the function in Y₁ over the given integration range using Simpson's Rule with N sub-intervals. You specify N; if N is odd, it uses N+1 sub-intervals.

Program: NUMINT/NUMINTCE calculates and displays (without graphics) the approximation to the integral of Y₁ over the given integration range using Simpson's Rule with 2N sub-intervals to ensure an even number. You specify N.

Activity 2 (Section 11.4.2) goes into more detail about and compares the Left-Endpoint Rule, the Trapezoidal Rule and Simpson's Rule.

Other program: The spinner program **SPNINTCE** does the graphics that NINTGRPH/ NINTGRCE does, controlled by on-screen commands (see the end of Section 11.4.2).

11.2.2   Gauss quadrature

Unlike the geometric methods of Section 11.2.1, Gauss quadrature methods do not have a simple geometric interpretation. The term Gauss quadrature covers several methods in which a definite integral is approximated as the weighted sum of n function values, with the weights w_i and the x values at which the function is evaluated, x_i, specified for a given method and a given order n;[16] the x_i are not evenly spaced, in general, unlike in the geometric methods. The geometric methods also fit into this category because their formulas follow the same format.

Reference: Hooper, M, Dharmasena, V and Cui, M., *Exploring Gaussian quadrature with students: Part 1 — A forgotten idea*, Australian Senior Maths Journal, 32 (2), 23–31 (2021).

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[16]These can be obtained from a number of sources. I used the website *keisan.casio.com* under *Professional/ Numerical integration*. This website also does the calculations.

1. Theory

A Gauss quadrature sum over the interval $[-1, 1]$ is of the form

$$\int_{-1}^{1} f(x)\, dx \;\approx\; \sum_{i=1}^{n} w_i f(x_i),$$

where the w_i are the specified weights, the x_i the specified x values at which the function is evaluated and n the number of these points (the order of the method).

For the general integral, over the interval $[a, b]$, a change of variables gives

$$\int_{a}^{b} f(x)\, dx \;\approx\; \frac{b-a}{2} \sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right).$$

2. Gauss-Legendre quadrature

Here the w_i and x_i are related to the Legendre functions P_n. The x_i for Gauss-Legendre quadrature of order n are the zeros of P_n: $P_n(x_i)=0$. The weights x_i are given by

$$w_i = \frac{2(1-x_i^2)}{\left(nP_{n-1}(x_i)\right)^2}.$$

Program: GLQUAD/GLQUADCE calculates the Gauss-Legendre quadrature of orders 9 and 11 for the function in Y₁ over the interval $[A, B]$, where you specify A and B. Comparison of the two results gives you some idea of the accuracy. The residual decimals are to be added to the first value to give an answer to 14 digits. This is a non-adaptive algorithm with a fixed number of points.

```
GAUSS-LEGENDRE INTEGRAL
   OF Y1 FROM A TO B

Y1=Xe^(-X)
A=0
B=4
ORDER 9:   0.9084218056
ORDER 11:  0.9084218056

                 [ENTER]
```

```
RESIDUAL DECIMALS

ORDER 9:  -4.37E-11
ORDER 11: -4.37E-11
```

Gauss-Legendre quadrature of orders 9 and 11

3. Gauss-Kronrod quadrature

Follows the same process as Gauss-Legendre quadrature but with different w_i and x_i.

Program: GAUSSQ11/GSQD11CE calculates the Gauss-Legendre quadrature and the Gauss-Kronrod quadrature of order 11 for the function in Y₁ over the interval $[A, B]$, where you specify A and B. The residual decimals are to be added to the first value to give an answer to 14 digits.

The Gauss-Kronrod quadrature here is a non-adaptive algorithm with a fixed number of points, unlike the calculator version (see below).

```
GAUSS INTEGRAL (ORDER 11)
    OF Y1 FROM A TO B

Y1=Xe^(-X)
A=0
B=4
G-LEG: 0.9084218056
G-KRO: 0.9084218056
                    [ENTER]
```

```
RESIDUAL DECIMALS

G-LEG:  -4.37ᴇ-11
G-KRO:  -4.368ᴇ-11
```

Gauss-Legendre and Gauss-Kronrod quadrature of order 11

4. fnInt: the TI-84 numerical integrator

From TI

The method used in the TI-84 family is Gauss-Kronrod integration. It is a good deal more sophisticated than commonly known methods like Simpson's Rule. A detailed discussion is much better left to the literature on this subject, which is voluminous. A few comments, however, may help the user in using this function.

The concept begins with Gaussian quadrature rules, which have the property that by sampling a function at n points, an integral can be estimated that would be exact if the function were a polynomial of degree $2n-1$. For comparison, the Trapezoidal Rule is of polynomial degree 1 and Simpson's Rule is of polynomial degree 3. But while Simpson's Rule gives an exact integral for cubic polynomials based on 3 points, a Gauss quadrature rule with 3 points will give the exact integral for polynomials up to order 5.

The next concept is to use a pair of Gauss quadrature rules of different order to provide both an integral result and a corresponding error estimate. The development of optimal extension pairs (pairs for which the lower-order sample points are a subset of the higher-order ones, so that extra function evaluations are avoided) was done by A.S. Kronrod in 1965. The use of these pairs allows the method to be 'adaptive'.

The method first calculates the integral and the error estimate for the whole interval. If the error is too large, the interval is bisected and the process repeated on each half. Anytime a segment passes with regard to its 'share' of the total error budget, that integral is added to the total integral. Other segments that don't pass are further divided. Note that this is quite different to some 'adaptive' methods that iterate with smaller intervals until the result changes by less than some tolerance. The method calculates the error from two integral estimates and only takes new function samples in rapidly changing regions that require it.

Use: The numerical-integration command in CLASSIC mode is of the form

$$\text{fnInt}\big(f(\text{X}), \text{X}, a, b\,[,\,tol]\big),$$

where X is the variable we are integrating the function f with respect to (can be any letter or θ), a and b are the integration limits and the optional argument *tol* is the tolerance (below left). The default value is 10^{-5}. In MATHPRINT mode (below right), you can't specify a tolerance.

```
fnInt(Y1,X,0,4,ᴇ-6)
              0.9084218056
```

```
∫⁴₀(Y1)dX
              0.9084218056
```

fnInt with $\text{Y1} = \text{X}e^{-\text{X}}$

The tolerance here is an 'absolute' error bound. This means the algorithm will quit when the absolute value of the difference in its two integrals is less than the tolerance you specify. Hence, it may not be wise to ask for a tolerance of 0.00001 if computing an integral that has a value of order $10,000$ because 10 accurate digits must be given. Another thing to watch out for is the fact the function is sampled a finite number of times on the first integration attempt. These points are not equally spaced and are in fact clustered toward the end points. However, it is not uncommon to have a function that is virtually constant or perhaps zero throughout much of the integration range. In this case, the integrator may quit early and fail to detect the existence of function behaviour in other regions.

An example of this is: $\text{fnInt}(e^{\wedge}(-T/10^{-6}), T, 0, 1)$. This may seem appropriate to an electrical engineer who wants to compute all the 'energy' in this waveform with a microsecond-order fall time. However, at x just greater than 0.0002 (the integration interval is 0 to 1), the value of this function becomes less than 10^{-999} and underflows to zero. So the algorithm returns a value of zero for this integral. An integration range of $(0, 0.0001)$ would be more appropriate for the problem and, because the integral is small, a *tol* of 10^{-10} is not excessive.

11.3   Comparison of methods

The table below gives values for $\int_0^4 xe^{-x}\,dx$ using the different methods with 11 points (function evaluations). Values are rounded to 6 significant digits.

LER	0.879810
RER	0.909115
MID	0.915361
TRP	0.894462
SIM	0.908006
GLE	0.908422
GKR	0.908422
fnInt	0.908422

The exact answer is $1 - 5e^{-4} = 0.908422$ to 6 significant digits. This answer and the last three in the table differ only in the 14th decimal place.

11.4 Activities

11.4.1 Rectangles, area and the definite integral

The TI-84/CE program NINTGRPH/NINTGRCE[17] illustrates graphically how the area under a graph can be approximated by the areas of rectangles. As the number of rectangles covering the area increases, we obtain a better approximation to the area. Here we approximate $\int_0^1 e^x\,dx$ by drawing rectangles (the sum of the areas is a *Riemann sum*).

- Put the function $f(x) = e^x$ in Y₁.

- Set a $\boxed{\text{window}}$ of $[0, 1, 0.2] \times [0, 3, 1]$.

- Run the program: press $\boxed{\text{prgm}}$, press the number against its name and press $\boxed{\text{enter}}$ to start the program.

- Set the integration limits $A = 0$ and $B = 1$.

- Set the number of rectangles $N = 5$.

- Choose the Left-Endpoint Rule. The program will plot the function and draw in 5 rectangles, each rectangle touching the curve at its top-left corner. In this case, the area of the rectangles clearly underestimates the area under the graph.

- Press $\boxed{\text{enter}}$ to see the area of the rectangles (LHSUM) as an approximation to the area under the graph.

- Press $\boxed{\text{enter}}$, set $N = 5$ again and choose the Right-Endpoint Rule. This time we obtain an overestimate of the area under the curve.

- Repeat the above steps, doubling the number of rectangles each time. Fill in the table below. Round your answers to 3 decimal places.

  The mean of the two estimates is equivalent to the Trapezoidal-Rule approximation to the area, a more accurate approximation for a given N than either the Left- or Right-Endpoint Rules.

N	LER	RER	Mean
5			
10			
20			
40			
80			

Your best estimate:

$\int_0^1 e^x\,dx \approx$

Compare this with the exact value if you know how to do the integral.

- When you've finished, select QUIT in the RULE menu.

---

[17]NINTGRPH is for the TI-84; NINTGRCE for the TI-84CE. You can also use the spinner version SPNUMINT/SPNINTCE: see page 79.

Teacher's notes

Answers   *rounded to 3 decimal places.*
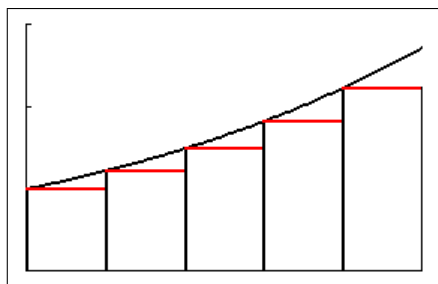
Left-Endpoint Rule
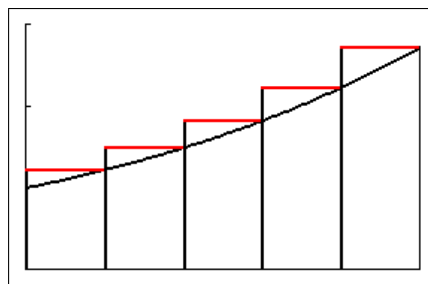
Right-Endpoint Rule

N	LER	RER	Mean
5	1.552	1.896	1.724
10	1.664	1.806	1.720
20	1.676	1.762	1.719
40	1.697	1.740	1.718
80	1.708	1.729	1.718

Best estimate is 1.718 from the MEAN column. The fact that two successive values give 1.718 indicates that this is probably the exact answer rounded to 3 decimal places,

The exact answer is $e-1=1.718$ rounded to 3 decimal places.

11.4.2   Approximating definite integrals

Modified from an UNSW Canberra Maths Lab, which is itself based on a lab in *Resources for Calculus, Volume 1: Learning by Discovery*, Anita Solow, editor, Mathematical Association of America Note 26, 1993.

In this lab, we shall be comparing several numerical approximations to

$$\int_0^1 \left(5x^4 - 3x^2 + 1\right) dx$$

with the exact answer obtained by algebraic integration. This will give us a feel for some of the methods of numerical integration, which we can then use for any function, including those which cannot be integrated algebraically.

Question 1   *Algebraic integration — the exact answer*

What is the exact value of this integral? You may not realise it, but you are using the *Fundamental Theorem of Calculus* to do this definite integral exactly.

Question 2   *The Left-Endpoint Rule*

One approach to numerical integration is to approximate the definite integral of $y = f(x)$ with $a \leqslant x \leqslant b$ by the sum of the areas of a number of rectangles under the curve. If the top left-hand corner of each rectangle touches the curve, we have the *Left-Endpoint Rule*; if the top right-hand corner of each rectangle touches the curve, we have the *Right-Endpoint Rule*.

As the number of rectangles in the interval $[a, b]$ gets larger and larger (covering the integration range $a \leqslant x \leqslant b$ with more and more, thinner and thinner rectangles), both rules give numbers closer and closer to the definite integral (exact answer).

(a) On Figure 1 (at the end of this lab), draw and shade in the rectangles for the Left-Endpoint-Rule approximation to the definite integral $\int_a^b f(x)\,dx$ with N = 4 (4 rectangles).

(b) Using your sketch in (a), explain why the Left-Endpoint Rule with 4 rectangles approximates the area under the graph as

$$h\big(f(x_0) + f(x_1) + f(x_2) + f(x_3)\big),$$

where $x_0 = a$, $x_4 = b$ and the width of each rectangle is $h = (b-a)/4$.

(c) Use the NINTGRPH/NINTGRCE program (instructions below)[18] to estimate the definite integral $\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ using the Left-Endpoint Rule with the number of rectangles N = 4. A suitable window is $[0, 1, 0.5] \times [0, 3, 1]$.

Note that the integrand here is positive, so that the definite integral corresponds to the area under the graph of f.

(d) Now use the program, doubling N until two successive answers from the Left-Endpoint Rule are the same when rounded to 2 decimal places. Write down the N value of the first of these two answers.

---

[18]NINTGRPH is for the TI-84; NINTGRCE for the TI-84CE. You can also use the spinner version SPNUMINT/SPNINTCE: see page 79.

Question 3   *The Trapezoidal Rule*

The Left-Endpoint and Right-Endpoint Rules approximate the area under a function by rectangles. In many cases, for example the function in Figure 1 with the rectangles you drew in, this is not a good approximation. We get a better approximation by using trapeziums: both top corners of each trapezium touch the curve.

(a) On Figure 2 (at the end of this lab), draw and shade in the 4 trapeziums (N=4), the total area of which approximates the definite integral $\int_a^b f(x)\,dx$.

The area of the trapezium in Figure 3 is $h(r+s)/2$. To see this result, split the trapezium into two regions — a triangle and a rectangle.
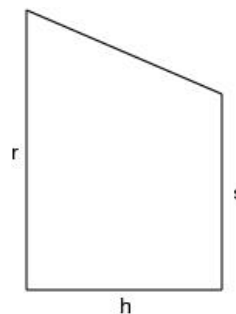
Figure 3

(b) Using your sketch in (a) and Figure 3, explain why the Trapezoidal Rule with 4 trapeziums approximates the area under the graph as

$$\frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right),$$

where $x_0 = a$, $x_4 = b$ and the width of each trapezium is $h = (b-a)/4$.

(c) Evaluate T_4, the Trapezoidal Rule with 4 trapeziums, as an estimate of the integral $\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ using NINTGRPH/NINTGRCE.
How does this result compare with the Left-Endpoint result and the exact answer?

(d) Now use the program, doubling N, the number of trapeziums, until two successive answers are the same when rounded to 2 decimal places. Write down the N value of the first of these two answers. Compare it with the Left-Endpoint value.

Question 4   *Simpson's Rule*

A picture of Simpson's Rule for which N=4 is given in Figure 4 (at the end of this lab). We want to estimate the area under the solid curve. We do this by a fitting parabola to each set of 3 successive points on the graph and adding up the areas under the parabolas.

The dashed line in Figure 4 shows two parabolas: one through $(x_0, f(x_0)),(x_1, f(x_1))$ and $(x_2, f(x_2))$; the other through $(x_2, f(x_2))$, $(x_3, f(x_3))$ and $(x_4, f(x_4))$.

(a) On Figure 4, shade the area calculated by Simpson's Rule as an approximation to the definite integral $\int_a^b f(x)\,dx$.

(b) Evaluate S_4, Simpson's Rule with 4 divisions of the integration interval, as an estimate of $\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ using NINTGRPH/NINTGRCE. The dotted lines are the two parabolas.

Compare your result with those from Questions 1 – 3.

(c) Now use the program, doubling N, the number of sub-divisions of the integration interval, until two successive answers are the same when rounded to 2 decimal places. Write down the N value of the first of these two answers.

Compare your result with those from Questions 2 and 3.

Question 5   *Comparing the methods*

Repeat your first (N = 4) and last calculations above for the three methods, this time keeping 5 decimal places. Put them in a summary table, together with the h value and the absolute value of the error $|E|$ for each entry (you know the exact answer).

The NUMINT/NUMINTCE program (no graphics) might be faster for this but note the N for Simpson's Rule (instructions below).

If $|E| = kh^m$, where k is a constant, find m for each method. *Hint*: Use the two sets of values of h and the corresponding errors to write down two equations for k and m; divide one equation by the other to obtain an equation for m. You'll need natural logs to isolate m. *Hint*: We are looking for integer values.

How many times do you have to double N in each method to improve the accuracy by a factor of 10?

What conclusions can you draw from your results regarding the different methods for estimating the definite integral? Which method would you choose to use? Why?

Calculator programs

These programs calculate approximate values for $\int_A^B f(X)\,dX$.

The number N is an input to the program.

NINTGRPH/NINTGRCE approximates the integral using one of five different rules **with N sub-divisions** of the interval [A, B] (N+1 if N is odd, for Simpson's Rule), and draws the corresponding approximations to the function on each subinterval.

NUMINT/NUMINTCE approximates the integral using the *Left-Endpoint Rule* (**L**), the *Right-Endpoint Rule* (**R**), the *Trapezoidal Rule* (**T**) and the *Midpoint Rule* (**M**), **all with N sub-divisions**, and *Simpson's Rule* (**S**) **with 2N sub-divisions** to ensure an even number of sub-divisions.

Use: Type the function to be integrated into Y₁.

- For NINTGRPH/NINTGRCE, first set a suitable window to display the function. Run the program and follow the prompts. Make sure B>A, otherwise things get mixed up. After the graph is plotted, press $\boxed{\text{enter}}$ to see the numerical approximation to the integral.

  When you've finished, select QUIT in the RULE menu.

- For NUMINT/NUMINTCE, run the program and follow the prompts.

  $\boxed{\text{on}}$ Quit stops the program.

Numerical-integration spinner

A spinner,[19] here the program SPNUMINT/SPNINTCE[20], allows you to vary on-screen the number of sub-divisions N and the method of each of five types of numerical integration, by pressing a key, and see the result replotted immediately. You specify the function to be integrated and the integration limits. The program contains the necessary information (in HELP) on how to vary the parameters.

The figures below show the main screen (where you set N and the rule) and each of the five rules for the function $f(x) = 1 + x\sin^2(3x)$, integrated over $0 \leqslant x \leqslant 3$, with N $= 8$. The rules are: the Left-Endpoint Rule (LER); the Right-Endpoint Rule (RER); the Midpoint Rule (MPR); the Trapezoidal Rule (TRP); and Simpson's Rule (SPR). The original window is $[0, 3, 1] \times [0, 4, 1]$ (Ymax is changed by the program to 5.2).

The EXACT value comes from the built-in integrator on the calculator *fnInt*.[21] All values rounded to 4 significant digits.

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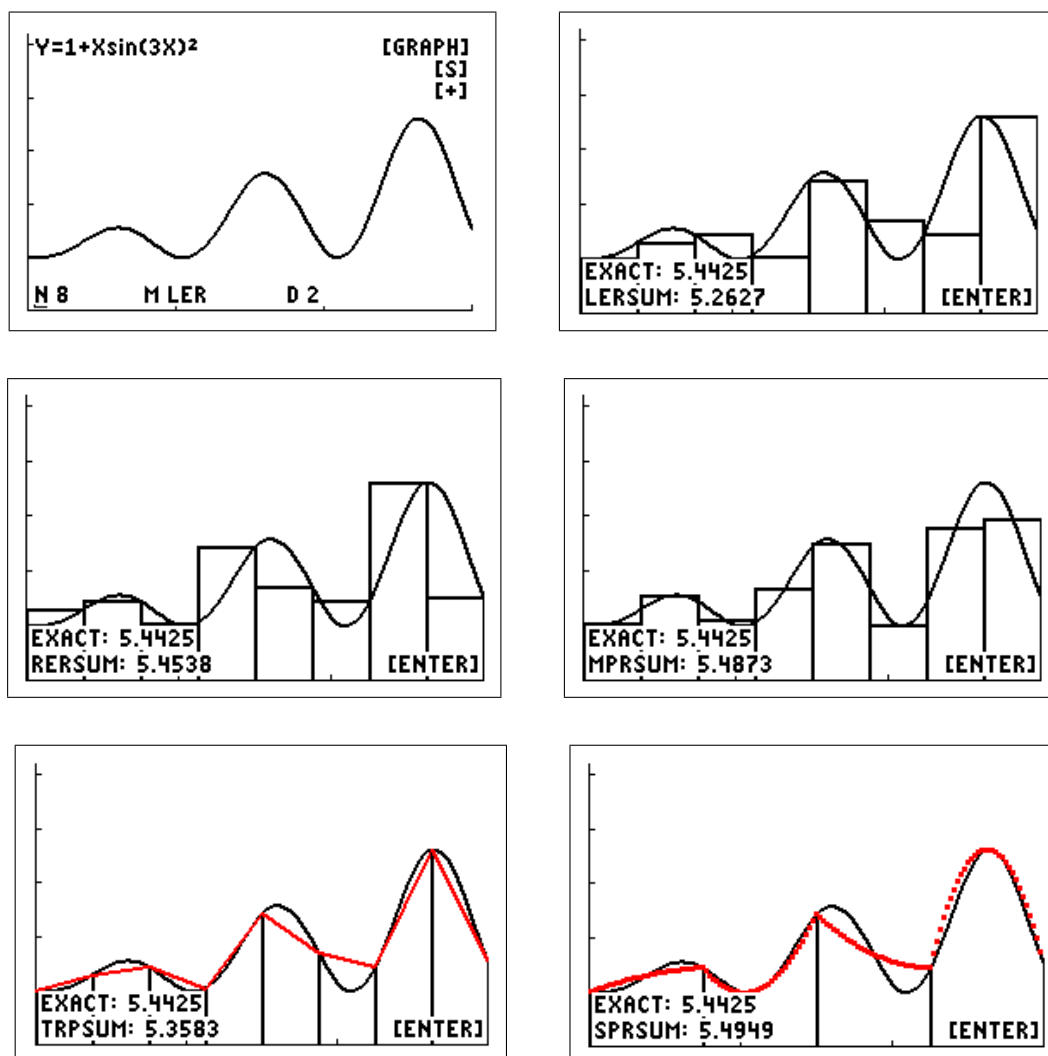
[19]from a feature in an earlier version of Excel used by Ed Staples and Margie Smith in SpinnaMaths to explore the effects of varying parameters in a variety of applications; sadly, SpinnaMaths is no longer available because of subsequent 'upgrades' to Excel.
[20]available at *www.XXX*
[21]The exact answer is $(379 - 18\sin(18) - \cos(18))/72$. On the calculator, this answer agrees with the fnInt answer to 13 significant digits.

Numerical Integration Lab Figures

The function in Figure 1 is not the function in Question 1.

Figure 1: Left-Endpoint Rule
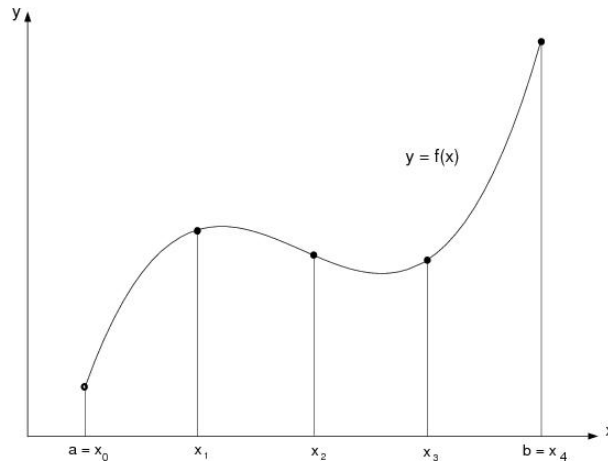
Figure 2: Trapezoidal Rule
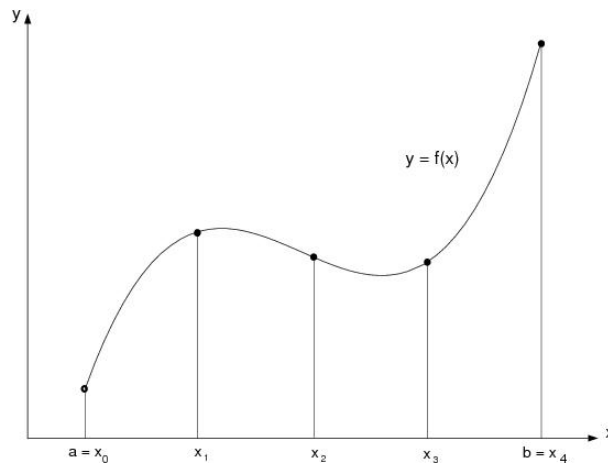
Figure 4: Simpson's Rule
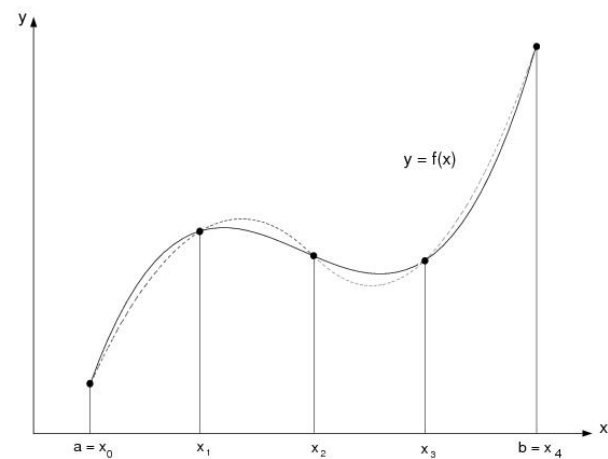
Teachers' notes

Question 1   *Algebraic integration — the exact answer*

$$\int_0^1 \left(5x^4 - 3x^2 + 1\right) dx = \left[x^5 - x^3 + x\right]_0^1 = 1.$$

Question 2   *The Left-Endpoint Rule*

(a) On Figure 1, draw and shade in the rectangles for the Left-Endpoint-Rule approximation to the definite integral $\int_a^b f(x)\, dx$ with $N = 4$.

Done here with the function in Question 1 using the NINTGRCE program (no shading). The horizontal red lines show the approximation to the function f on each sub-interval.
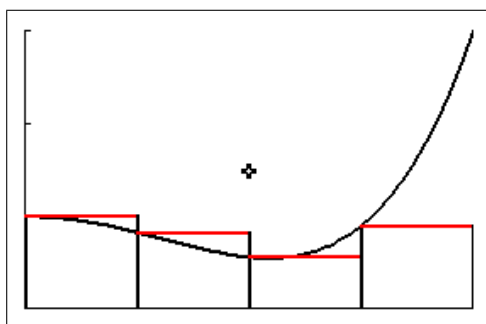
window $[0, 1, 0.5] \times [0, 3, 1]$

(b) Using your sketch in (a), explain why the Left-Endpoint Rule with 4 rectangles approximates the area under the graph as

$$h\big(f(x_0) + f(x_1) + f(x_2) + f(x_3)\big),$$

where $x_0 = a$, $x_4 = b$ and the width of each rectangle is $h = (b - a)/4$.

The formula is just the sum of the areas of the 4 rectangles in (a) with the common width h factored out.

(c) Use the NINTGRPH/NINTGRCE program to estimate $\int_0^1 (5x^4 - 3x^2 + 1)\, dx$ using the Left-Endpoint Rule with $N = 4$. A suitable window is $[0, 1, 0.5] \times [0, 3, 1]$.

Note that the integrand here is positive, so that the definite integral corresponds to the area under the graph of f.

See the figure in (a).

The Left-Endpoint Rule with $N = 4$ gives $\displaystyle\int_0^1 \left(5x^4 - 3x^2 + 1\right) dx \approx 0.822$.

(d) Now use the program, doubling N, the number of rectangles, until two successive answers from the Left-Endpoint Rule are the same when rounded to 2 decimal places. Write down the N value of the first of these two answers.

N	LER
4	0.82
8	0.89
16	0.94
32	0.97
64	0.98
128	0.99
256	1.00
512	1.00

The required N value is therefore 256. The last two values are the same as the exact answer rounded to 2 decimal places.

Question 3   *The Trapezoidal Rule*

(a) On Figure 2, draw and shade in the 4 trapeziums (N$=$4), the total area of which approximates the definite integral $\int_a^b f(x)\,dx$.

Again done here with the function in Question 1 but without shading. The red straight lines show the approximation to the function f on each sub-interval.
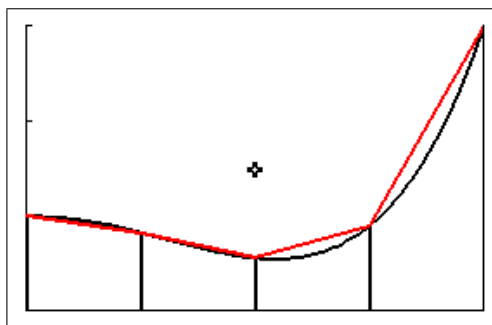
window $[0, 1, 0.5] \times [0, 3, 1]$

(b) Using your sketch in (a) and Figure 3, explain why the Trapezoidal Rule with 4 trapeziums approximates the area under the graph as

$$\frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right),$$

where $x_0 = a$, $x_4 = b$ and the width of each trapezium is $h = (b-a)/4$.

The total area of the four trapeziums each of width $h=0.25$ is, using the given formula,

$$h\,\frac{f(0)+f(x_1)}{2} \;+\; h\,\frac{f(x_1)+f(x_2)}{2} \;+\; h\,\frac{f(x_2)+f(x_3)}{2} \;+\; h\,\frac{f(x_3)+f(x_4)}{2}$$

$$= \;\frac{h}{2}\big(f(x_0)\;+\;2f(x_1)\;+\;2f(x_2)\;+\;2f(x_3)\;+\;f(x_4)\big).$$

(c) Evaluate T_4, the Trapezoidal Rule with 4 trapeziums, as an estimate of the definite integral $\int_0^1 (5x^4-3x^2+1)\,dx$ using NINTGRPH/NINTGRCE.

How does this result compare with the Left-Endpoint result and the exact answer?

$T_4=1.07$, considerably closer to the exact answer 1 than the Left-Endpoint Rule value with $N=4$ of 0.82.

(d) Now use NINTGRPH/NINTGRCE, doubling N, the number of trapeziums, until two successive answers are the same when rounded to two decimal places. Write down the N value of the first of these two answers. Compare it with the Left-Endpoint N value.

N	TRP
4	1.07
8	1.02
16	1.00
32	1.00

The required N value is therefore 16, compared with the much larger value of 256 for the Left-Endpoint Rule. The last two values are the same as the exact answer rounded to 2 decimal places.

Question 4   *Simpson's Rule*

(a) On Figure 4, shade the area calculated by Simpson's Rule as an approximation to the definite integral $\int_a^b f(x)\,dx$.

Shade below the dotted curves.

(b) Evaluate S_4, Simpson's Rule with 4 sub-divisions of the integration interval, as an estimate of $\int_0^1 (5x^4-3x^2+1)\,dx$ using NINTGRPH/NINTGRCE.

The red dotted lines show the two parabolic approximations to the function f on each two sub-intervals.
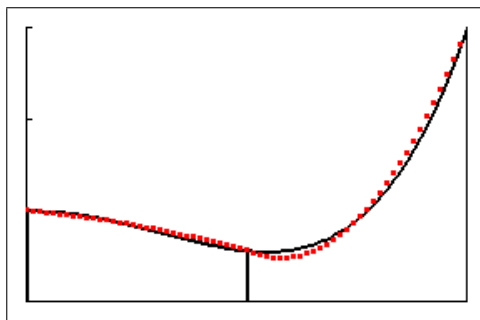
window $[0,1,0.5]\times[0,3,1]$

Compare your result with those from Questions $1-3$.

$S_4 = 1.003$, considerably closer to the exact answer 1 than the other two values.

(c) Now use NINTGRPH/NINTGRCE, doubling the number N of divisions of the integration interval each time, until two successive answers are the same when rounded to two decimal places. Write down the N value of the first of these two answers.

Compare it with the rectangle and trapezium values.

N	SMP
4	1.00
8	1.00

The required N value is therefore 4, compared with 16 for the Trapezoidal Rule and the much larger value of 256 for the Left-Endpoint Rule. The two values here are the same as the exact answer rounded to 2 decimal places.

Question 5   *Comparing the methods*

If $|E| = kh^m$, where k is a constant, find m for each method.

Summary table: $|E|$ is $|\text{Value} - 1|$.

Method	N	h	Value (5DP)	$|E|$ (5DP)
Left Endpoint	4	0.25	0.82227	0.17773
Left Endpoint	256	0.0039	0.99611	0.00389
Trapezoidal	4	0.25	1.07227	0.07227
Trapezoidal	16	0.0625	1.00455	0.00455
Simpson	4	0.25	1.00260	0.00260
Simpson	8	0.125	1.00016	0.00016

Assume error $|E| = kh^m$, where k is a constant. Therefore, if we have values E_1, h_1, E_2 and h_2 for a method,

$$\frac{E_1}{E_2} = \frac{h_1^m}{h_2^m} = \left(\frac{h_1}{h_2}\right)^m.$$

Taking natural logs of both sides and solving for m, we have

$$m = \frac{\ln\left(\frac{E_1}{E_2}\right)}{\ln\left(\frac{h_1}{h_2}\right)}.$$

Substituting in the two values for E and h for each method, we get $m \approx 1$ for the Left-Endpoint Rule, $m \approx 2.0$ for the Trapezoidal Rule and $m \approx 4.0$ for Simpson's Rule.

How many times do you have to double N in each method to improve the accuracy by a factor of 10?

If you double N, you halve h. Therefore, for the Left Endpoint Rule, you reduce the error by 0.5^1, i.e. by a factor of 2. For the Trapezoidal Rule, you reduce the error by 0.5^2, i.e. by a factor of 4. For Simpson's Rule, you reduce the error by 0.5^4, i.e. by a factor of 16.

To achieve an improvement in accuracy of 1 decimal place, you have to reduce the error by a factor of 10: therefore, using the Left Endpoint Rule you have to halve h four times $(0.5^3 = 0.125 > 0.1$, not enough; $0.5^4 = 0.0625 < 0.1)$, for the Trapezoidal Rule twice $((0.5^2)^2 = 0.0625 < 0.1)$ and, for Simpson's Rule only once $((0.5^4)^1 = 0.0625 < 0.1)$.

What conclusions can you draw from your results regarding the different methods for estimating the definite integral? Which method would you choose to use? Why?

For a given N or h, Simpson's Rule gives the most accurate approximation to the definite integral. To calculate an approximation to a given accuracy, it will therefore be the fastest of the three methods.

12   Taylor Series

12.1   Introduction

Values of polynomial functions can be found by performing a finite number of additions and multiplications but other special functions such as $\ln(x)$, e^x, $\cos(x)$ and $\tan(x)$ cannot be evaluated as easily. Many functions can be approximated by polynomials, and a polynomial, instead of the original function, can be used for computations when the difference between the actual function and the polynomial approximation is sufficiently small. Calculators and computers use polynomials of some sort to calculate values of special functions.

Graphics calculators are ideal for this topic because they show the result graphically of all the algebra in calculating the Taylor polynomials and how the polynomials converge or otherwise to the function they are approximating. The interval of convergence of a series can also usually be found from the graphs.

The exercises and problems here go from the basic calculation up to several interesting approximate models for physical phenomena.

12.2   Approximating functions by polynomials

Various methods can be employed to approximate a given function by polynomials. One of the most widely used is the method of Taylor polynomials, named in honour of the English mathematician Brook Taylor (1685 – 1731), who introduced them in 1715. Taylor polynomials about $x = 0$ are also called Maclaurin polynomials after the Scottish mathematician Colin Maclaurin (1698 – 1746), who made extensive use of this special case of Taylor series in the 18th century.

12.2.1   Linear approximation

The tangent line at a point on the curve of a function is a *linear approximation* to the function at that point; it has the *same value* and the *same first derivative* (slope) there as the function.

Example 1: Find the equation of the tangent line (L in the figure below) as a linear approximation to $f(x) = e^x$ near $x = 0$

Use the tangent-line approximation to estimate $e^{0.1}$.

Compare with the calculator value (presumably reasonably accurate).
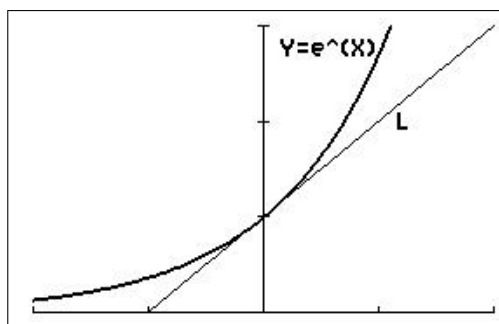
window $[-2, 2, 1] \times [-0.3, 3, 1]$

The function

$f(0) = e^0 = 1.$ $f'(x) = e^x$, so that $f'(0) = 1.$

The linear approximation

The general equation of a straight line is $y(x) = a_0 + a_1 x$, where a_0 and a_1 are constants.

The tangent has the same value as f at $x = 0$. Therefore,

$$y(0) = a_0 = f(0) = 1.$$

The tangent has the same slope as the graph of f at $x = 0$: $y'(x) = a_1$. Therefore,

$$y'(0) = a_1 = f'(0) = 1.$$

Substituting for a_0 and a_1 in the equation of the straight line gives the linear approximation to $f(x) = e^x$ centred at $x = 0$ as

$$y(x) = 1 + x.$$

Therefore, $f(0.1) = e^{0.1} \approx y(0.1) = 1.1.$

The exact value is $f(0.1) = e^{0.1} = 1.105$, to 4 significant digits.

12.2.2   Quadratic approximation

Suppose we want a more accurate way of approximating $f(x) = e^x$ near $x = 0$. We will require that, at $x = 0$, the curve $f(x) = e^x$ and the approximating curve have *the same value, the same first derivative* (slope) and also the same rate of bending (concavity), that is *the same second derivative.*

The simplest type of function we might use for this approximation is a quadratic, with general formula[22]

$$P_2(x) = a_0 + a_1 x + a_2 x^2, \tag{1}$$

where we must determine the values of a_0, a_1 and a_2 to fit the quadratic to our function.

At $x = 0$ we want P_2 and f to have the same value, so we want

$$P_2(0) = f(0). \tag{2}$$

We want P_2 and f to have the same first derivative at $x = 0$, so

$$P_2'(0) = f'(0). \tag{3}$$

and we also want P_2 and f to have the same second derivative at $x = 0$, so

$$P_2''(0) = f''(0). \tag{4}$$

These three conditions will determine a_0, a_1 and a_2.

From Eq. (1),

$$P_2(0) = a_0 = f(0) \quad \text{from Eq. (2).}$$
$$\therefore \quad a_0 = f(0).$$

---

[22]P for polynomial, 2 for degree 2. The linear approximation in Example 1 is $P_1(x)$.

Differentiating Eq. (1),

$$P_2'(x) = a_1 + 2a_2x.$$

$$\therefore\ P_2'(0) = a_1 = f'(0) \qquad \text{from Eq. (3) .}$$

$$\therefore\ a_1 = f'(0).$$

Continuing,

$$P_2''(x) = 2a_2$$

$$\therefore\ P_2''(0) = 2a_2 = f''(0) \qquad \text{from Eq. (4).}$$

$$\therefore\ a_2 = \frac{f''(0)}{2}.$$

Therefore,

$$P_2(x) = a_0 + a_1x + a_2x^2$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

This is a quadratic approximation to any function f centred at $x = 0$, provided that the function has finite first and second derivatives at $x=0$.

Exercise 1   *Solutions to the Exercises are on page 103.*

(a) Use the general result above to find the quadratic approximation (Q in the figure below) to $f(x)=e^x$ centred at $x=0$.
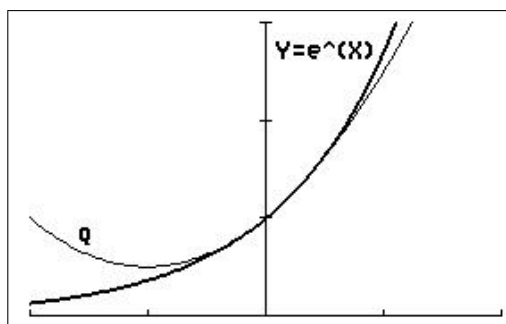
window $[-2, 2, 1]\times[-0.3, 3, 1]$

(b) Use the approximation to estimate $e^{0.1}$.

Compare with the calculator value and $P_1(0.1)$ (Example 1).

12.3 Taylor polynomials

12.3.1 Centred at $x=0$

As a rule, over a given interval the quadratic function will be a better approximation than the linear function. However, the figure on the previous page shows that, even though we have matched up the function and the quadratic in terms of their values, slopes and concavity *at the point* $x=0$, they still bend away from one another for x *well away from* $x=0$. We can improve this by using approximating polynomials of higher degree.

Suppose that we want to approximate a function $f(x)$ near $x=0$ by a polynomial of degree n, that is,

$$f(x) \approx P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_{n-1} x^{n-1} + a_n x^n. \tag{5}$$

We now need to find the values of the constants $a_0, a_1, \ldots, a_n$. To do this, we require that the values of the function and all of its n derivatives agree with those of the polynomial at $x = 0$. Polynomials which have this property are called **Taylor polynomials**, in this case Taylor polynomials centred at $x=0$.

Taylor polynomial of degree n for $f(x)$ centred at $x=0$

$$f(x) \approx P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Note: $P_n(x)$ is just $P_{n-1}(x)$ plus an extra term in x^n.

Example 2: Use the general formula to find the nth-degree Taylor polynomial centred at $x=0$ for $f(x)=e^x$.

For $f(x)=e^x$, $f(0)$ and all the derivatives evaluated at $x=0$ are equal to 1. Therefore,

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Plot f and $P_0 - P_4$ on your calculator. Y1, Y2, ... are in the $\boxed{\text{vars}}$ Y-VARS menu.
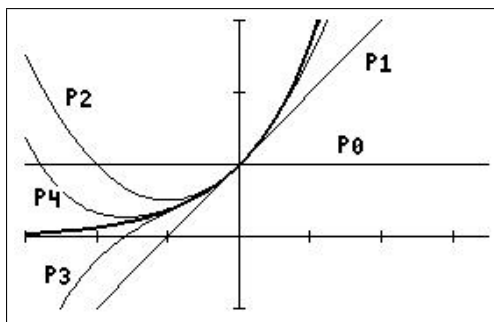
window $[-3, 3.5, 1] \times [-1, 3, 1]$;
the graph of f is the bold line

Use each polynomial to approximate $e^{0.2}$, and compare with the calculator value. Underline the digits that are correct in each approximate value.

$e^{0.2} \approx P_0(0.2) = \underline{1}$

$e^{0.1} \approx P_1(0.2) = \underline{1.2}$

$e^{0.2} \approx P_2(0.2) = \underline{1.22}$

$e^{0.2} \approx P_3(0.2) = \underline{1.221}\dot{3}$

$e^{0.1} \approx P_4(0.2) = \underline{1.2214}$

From the calculator, $e^{0.2} = 1.221403$ to 7 significant digits.

Exercise 2: Derive the result in the box on the previous page.

Hint: Repeat and extend the process begun in Section 12.2.2, but starting with Eq. (5) for P_n instead of Eq. (1) for P_2.

Once you have established a pattern in the coefficients, you may write $\vdots$, then the nth term.

Exercise 3: Find the Taylor polynomials centred at $x = 0$ for $f(x) = \sin(x)$, up to degree 11 (look for a pattern in the first few coefficients).

Plot P_1 and P_3 on the relevant graphs below; the window is $[-6, 6, 1] \times [-4, 4, 1]$.

Use each polynomial up to P_{11} to approximate $\sin(0.2)$, and compare with the calculator value. Underline the digits that are correct in each approximate value.
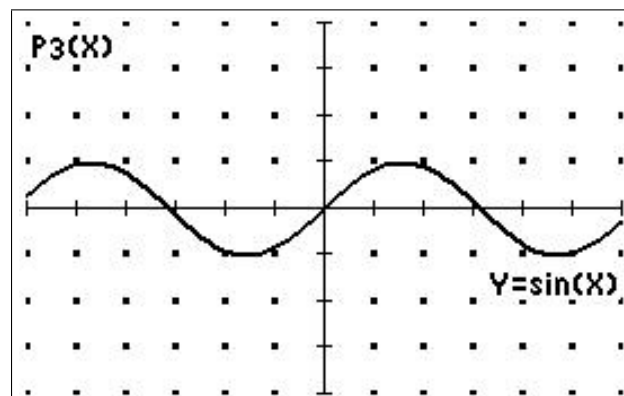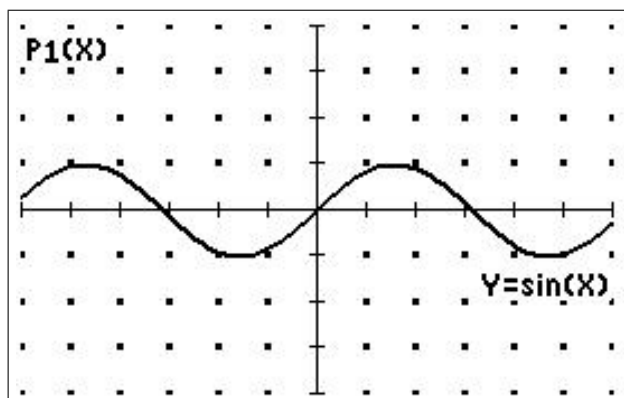
12.3.2   Centred at $x = c$

If instead of approximating a function $f(x)$ near $x = 0$, we want to approximate it near $x = c$, for the functions considered so far we may need a large number of terms in the Taylor polynomial about $x = 0$; the further c is from the origin, the more terms will be required. It makes more sense to use the Taylor polynomial centred at $x = c$.

Suppose we know the values of f and its derivatives at $x = c$. We want to find a polynomial $P_n(x)$ which is a good approximation to $f(x)$ for values of x close to c.

As the expression $x - c$ tells us how close x is to c, we use it to construct the polynomials approximating f at $x = c$. We write our general polynomial as

$$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n.$$

If we require the derivatives of the approximating polynomial and the original function to agree at $x = c$, we obtain the following result.

Taylor polynomial of degree n for $f(x)$ centred at $x = c$

$$f(x) \approx P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Example 3: Find the Taylor polynomials centred at $x = 1$ for $f(x) = \ln(x)$ up to degree 5.

$$P_5(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5.$$

$f(x) = \ln(x) \qquad f(1) = 0.$

$f'(x) = \dfrac{1}{x} \qquad f'(1) = 1.$

$f''(x) = -\dfrac{1}{x^2} \qquad f''(1) = -1.$

$f'''(x) = \dfrac{2}{x^3} \qquad f'''(1) = 2.$

$f^{(4)}(x) = -\dfrac{6}{x^4} \qquad f^{(4)}(1) = -6.$

$f^{(5)}(x) = \dfrac{24}{x^5} \qquad f^{(5)}(1) = 24.$

$$\therefore \; P_5(x) = (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4 + \frac{24}{5!}(x-1)^5$$

$$= x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5}.$$

Plot f and $P_1 - P_5$ on your calculator using a window $[-1, 4, 1] \times [-4, 4, 1]$.
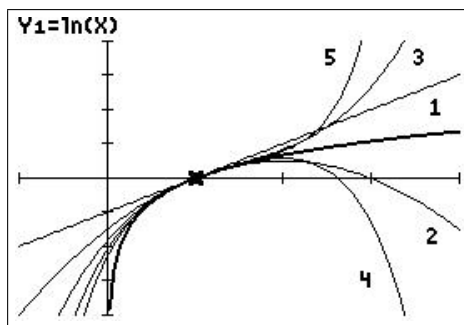
Use the successive polynomials to approximate $\ln(1.2)$, and compare with the calculator value. Underline the digits that are correct in each approximate value.

From the calculator, $\ln(0.2) = 0.1823215568$ to 10 significant digits.

$\ln(1.2) \approx P_1(1.2) = 0.2$

$\ln(1.2) \approx P_2(1.2) = \underline{0.18}$

$\ln(1.2) \approx P_3(1.2) = \underline{0.182}\dot{6}$

$\ln(1.2) \approx P_4(1.2) = \underline{0.1822}\dot{6}$

$\ln(1.2) \approx P_5(1.2) = \underline{0.1823}30\dot{6}$

Exercise 4: Find the fifth-degree Taylor polynomial for $f(x) = \dfrac{1}{x}$ centred at $x = 3$.

Plot f and $P_1 - P_5$ on your calculator using a window $[-1, 9, 1] \times [-2, 2, 1]$.

12.4   Taylor series

12.4.1   Centred at $x = 0$

We have seen how to approximate a function near a point by Taylor polynomials.

Example 4: The first few Taylor polynomials for $f(x) = \cos(x)$ centred at $x = 0$ are

$$\cos(x) \approx P_0(x) \;\; = \;\; 1$$

$$\cos(x) \approx P_2(x) \;\; = \;\; 1 - \frac{x^2}{2!}$$

$$\cos(x) \approx P_4(x) \;\; = \;\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\cos(x) \approx P_6(x) \;\; = \;\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\cos(x) \approx P_8(x) \;\; = \;\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}.$$

Note the $\approx$ sign. Each polynomial is a better approximation to $\cos(x)$ than the previous one.


```
■\Y₁Bcos(X)
■\Y₂B1
■\Y₃BY₂-X²/2!
■\Y₄BY₃+X^4/4!
■\Y₅BY₄-X^6/6!
■\Y₆BY₅+X^8/8!
■\Y₇BY₆-X^10/10!
■\Y₈BY₇+X^12/12!
■\Y₉BY₈-X^14/14!
```
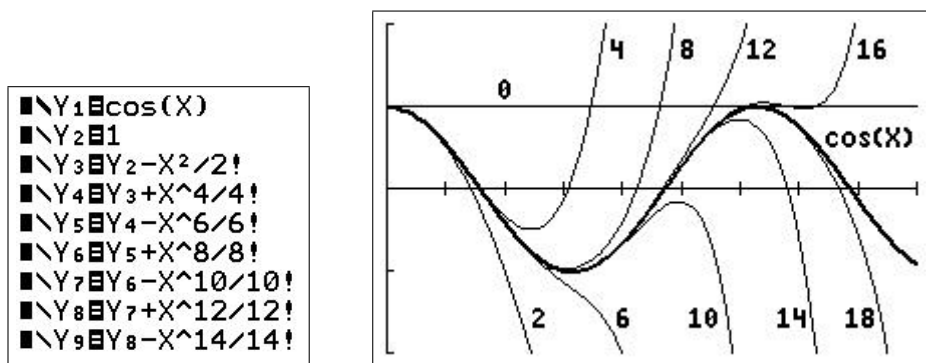
window $[0, 9, 1] \times [-2, 2, 1]$[23]

We are therefore tempted to write

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots ,$$

indicating that the cosine function is **equal** to this "infinite-degree" polynomial. This infinite sum is called the **Taylor series** for $\cos(x)$, centred at $x = 0$.

Note that $\cos(x)$ and the $P_n(x)$ here are all even functions: they are symmetric about the y axis.

---

[23]I plotted up to P_{16} (in Y0), that is 10 functions, the maximum number in $\boxed{\text{y=}}$, then stored the graph in Pic 1. I then plotted P_{18} and recalled Pic 1. The Store and Recall commands are in the $\boxed{\text{draw}}$ STO menu.

What do we mean by such an infinite sum? You may already have some intuition about what it means, for it can be interpreted in exactly the same way we interpret a more familiar statement like

$$\frac{1}{3} = 0.333333\cdots.$$

This means that $1/3$ is the *limit* of successive finite decimals $0.3, 0.33, 0.333, 0.3333, \ldots$.

Similarly, for any x, the value of $\cos(x)$ is the *limit* of the values of successive finite Taylor polynomials. We say that the Taylor series *converges* to $\cos(x)$ for all x, because for all x, the values

$$P_0(x),\ P_1(x),\ P_2(x), \cdots, P_n(x), \cdots$$

converge (get closer and closer to) to the value $\cos(x)$ as $n \to \infty$.

For example, at $x = 1$,

$$
\begin{aligned}
P_0(1) &= 1 \\
P_2(1) &= 0.\underline{5} \\
P_4(1) &= 0.\underline{54}16 \\
P_6(1) &= 0.\underline{54}027 \\
P_8(1) &= 0.\underline{5403025}794 \\
P_{10}(1) &= 0.\underline{5403023}028 \\
&\vdots \\
P_{18}(1) &= 0.\underline{54030230586}815 \\
&\vdots \\
\cos(1) &= 0.54030230586184 \quad \text{(to 14 significant digits)},
\end{aligned}
$$

and thus we see that $P_n(1)$ gets closer and closer to $\cos(1)$ as $n \to \infty$.

Some common Taylor series centred at $x = 0$

For all x,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + (-1)^n \frac{x^{2n-1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$$

In the last two series, the first term corresponds to $n = 0$. Note that $0! = 1$ by definition.

12.4.2   Centred at $x=c$

Any function f, all of whose derivatives exist at $x=c$, has a Taylor series centred at $x=c$.
Assuming that the series converges to $f(x)$, we have the following result,

> **Taylor series for $f(x)$ centred at $x=c$**
>
> $$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Example 5: Find the Taylor series centred at $x=1$ for $f(x)=e^x$.

As $f'(x)=f(x)$, we have $f(1)=f'(1)=f''(1)=f'''(1)=f^{(4)}(1) = \cdots = e$.

Therefore,

$$
\begin{aligned}
e^x &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \ldots \\
&= e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \cdots + \frac{e}{n!}(x-1)^n + \ldots \\
&= e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \frac{(x-1)^n}{n!} + \cdots\right) \\
&= e\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}.
\end{aligned}
$$

```
■\Y₁⊟e^(X)
■\Y₂⊟e
■\Y₃⊟Y₂+e(X-1)
■\Y₄⊟Y₃+e(X-1)²/2!
■\Y₅⊟Y₄+e(X-1)³/3!
■\Y₆⊟Y₅+e(X-1)^4/4!
■\Y₇=
■\Y₈=
■\Y₉=
```

Y6=Y5+e(X-1)^4/4!

X=2.25   Y=9.4011671
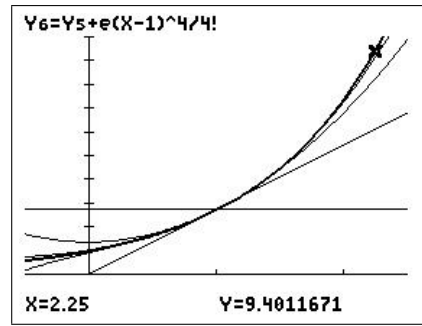
window $[-0.5, 2.5, 1] \times [0, 10, 1]$;
the cursor is on P_4

Exercise 5: Find the Taylor series centred at $x=\pi/2$ for $f(x)=\cos(x)$.

12.4.3 A neat method

Example 6: Find the Taylor series for $f(x) = e^{x^2}$ about $x = 0$.

We have, from the box on page 94,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Therefore, replacing x with x^2,

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \cdots + \frac{x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Clearly, this method is quicker than the standard method (see Exercise 7). Keep an eye out for other cases, especially if the standard method looks complicated.

Exercise 6: Find the Taylor series for $f(t) = \dfrac{\sin(t)}{t}$ centred at $t = 0$.

Exercise 7: Use the <u>standard method</u> to find the first three non-zero terms in the Taylor series for $f(x) = e^{x^2}$ centred at $x = 0$. Plot together with f. Compare the calculations here with those in Example 6.

12.4.4 Interval of convergence

Example 7: Let's look at the graph of $f(x) = 1/(1+x^2)$ and its successive Taylor polynomials centred at $x = 0$ (figure below center),

$$P_n(x) = 1 - x^2 + x^4 - x^6 + \cdots \pm x^n.$$

with $n = 2, 4, 6, 8, 10, 12, 14, 16, 200, 202.$[24] As the graphs of all the functions are symmetric about the y axis (*why is this?*), we only draw the them for positive x. There is also a vertical line drawn at $x = 1$.
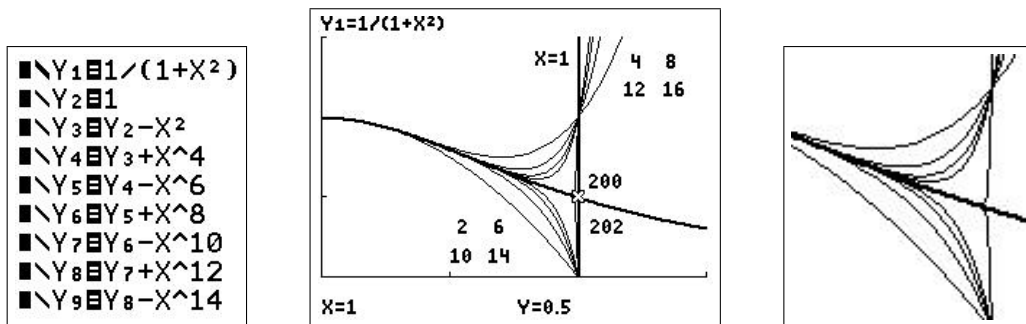
window $[0, 1.5, 0.5] \times [0, 1.5, 0.5]$

The figure above right shows the same plot, magnified, and with no vertical line at $x = 1$; this shows P_{200} and P_{202} (the almost vertical lines) more clearly.

---

[24]See the footnote on page 93. $P_{200}(x)$ is programmed as $Y_2 = \text{sum}(\text{seq}((-1)^N * X^{(2N)}, N, 0, 100, 1))$. *sum* is in the $\boxed{\text{LIST}}$ MATH menu, *seq* in the $\boxed{\text{LIST}}$ OPS menu. As the calculator has to sum a 101-term series for each point on the graph, it takes a while to plot the whole curve. Setting Xres $= 3$ in $\boxed{\text{window}}$ (plotting every third point) speeds up the process.

It appears that the graphs of the Taylor polynomials $P_n(x)$ approach the graph of $1/(1+x^2)$ (the bold curve) very nicely *as long as* $x < 1$. If $x > 1$, it looks like there is no convergence, no matter how many terms in the Taylor series we add. In fact, when $x > 1$, the larger the n, the worse the approximation P_n is to the function.

We can therefore write

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \qquad \text{for } |x| < 1,$$

where the restriction on x is *essential* if we want to use the $=$ sign.

We say that the interval $-1 < x < 1$ is the **interval of convergence** (IC) of the Taylor series for $1/(1+x^2)$ centred at $x = 0$.

Some Taylor series, such as those for $\sin(x)$, $\cos(x)$ and e^x, converge for *all* values of x, that is their IC is $(-\infty, \infty)$. Other Taylor series, such as those for $1/(1+x^2)$ and $\ln(x)$, have *finite* ICs.

Exercise 8: From the Taylor polynomials for $f(x) = \ln(x)$ centred at $x = 1$ in Example 3, write down the Taylor series and determine its IC by graphing the function and its first few Taylor polynomials.

Check by evaluating $P_n(1.2)$ and $P_n(2.2)$ for the first few values of n and comparing them with the values of $\ln(1.2)$ and $\ln(2.2)$.

Exercise 9: Find the Taylor series for $f(x) = \dfrac{1}{1-x}$ centred at $x = 0$ and determine its IC.

Exercise 10: Find the Taylor series for the function

$$f(x) = \begin{cases} \cos(x) & |x| \leqslant \pi \\ -1 & |x| > \pi \end{cases}$$

centred at $x = 0$. What is its IC?

Plot the function as $Y_1 = \cos(X)\big(\mathrm{abs}(X) \leqslant \pi\big) - 1\big(\mathrm{abs}(X) > \pi\big)$. The logical expressions in brackets evaluate to 1 if the expression is true and 0 if the expression is false.

Exercise 11

(a) Find the Taylor series and its IC for $f(x) = \ln(1+x)$ centred at $x = 0$.

(b) The Taylor series for $\ln(x)$ about $x = 1$ has an IC of $0 < x < 2$ (Example 3) and, as you found in (a), the series for $f(x)$ about $x = 0$ also has a very restricted IC. Neither is therefore suitable for finding $\ln(x)$ over its whole domain $x > 0$.

   Find the Taylor series centred at $x = 0$ for $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$ and its IC.
   Hint: use a log law and previous results.

(c) As x ranges across the IC you found in (b), what values does $g(x) = \dfrac{1+x}{1-x}$ take?

(d) Hence write out an algorithm to find the natural log of any number $x > 0$.

   Plot the difference between $\ln(x)$ and $P_5(x)$, and the difference between $\ln(x)$ and $P_{10}(x)$, for $0 < x < 20$.

12.5   Programs

Available at *www.XXX*

12.5.1   Taylor polynomials #1 — TAYLOR1/TAYLR1CE

Fits Taylor polynomials of degree $K = 1$, 2 or 3 about $X = C$ to the function Y_1, using numerical differentiation to evaluate the coefficients.

Use: Type the function into Y_1, run the program and follow the prompts. The graphs are displayed in $\boxed{\text{trace}}$ mode. Press $\boxed{\text{enter}}$ to continue. $\boxed{\text{on}}$ Quit stops the program.

Notes

  1. After you stop the program, press $\boxed{\text{graph}}$ to see the last plot. You can then use $\boxed{\text{trace}}$ and $\boxed{\text{zoom}}$ to explore the curves. The function and the three polynomials are stored in $Y_1 - Y_4$, respectively, so you can turn them on and off manually.

  2. The polynomials are given by

$$P_1(x) \ = \ Y_1(c) \ + \ Y_1'(c)(x-c)$$

$$P_2(x) \ = \ P_1(x) \ + \ \frac{Y_1''(c)}{2!}(x-c)^2$$

$$P_3(x) \ = \ P_2(X) \ + \ \frac{Y_1'''(c)}{3!}(x-c)^3$$

  3. The program uses nDeriv to work out the first derivative and nDeriv(nDeriv...) to work out the second derivative. nDeriv 3 times does not work and would be inaccurate anyway. The third derivative of a function f is given more accurately by

$$f'''(c) \ = \ \frac{f(c+2\Delta x) - 2f(c+\Delta x) + 2f(c-\Delta x) - f(c-2\Delta x)}{2(\Delta x)^3}.$$

Example: Try $Y_1 = e^{\wedge}X$ and $C = 1$. Do you get reasonable approximations to the function?

12.5.2   Taylor polynomials #2 — TAYLOR2/TAYLR2CE

Fits Taylor polynomials of degree $1 - 10$ about $X = C$ to function Y_1. Requires values for the up to ten derivatives of Y_1 at $X = C$.

Use: Type the function into Y_1, run the program and follow the prompts. The polynomials are a bit slow to plot, as the calculator has to evaluate a 10th-degree polynomial even if you specify fewer derivatives. Press $\boxed{\text{enter}}$ to continue.

Example: Continue the example $Y_1 = e^{\wedge}X$. All the derivatives are then $e^{\wedge}X$, so the values you input are all e. You can type it in in this form $\left(\boxed{\text{2nd}} \boxed{\div} \right)$.

12.6   Problems

Solutions to the problems are on page 115.

If the function is given, check your answer by plotting the function and Taylor polynomial(s).

1. Find the Taylor polynomials of degree $n=2,3,4$, centred at $x=0$, for $f(x)=\sqrt{1+x}$.

2. Find the Taylor polynomials of degree $n=3,4$, centred at $x=0$, for
$f(x) = \arctan(x) \equiv \tan^{-1}(x)$. Note that $f'(x) = 1/(1+x^2)$.

3. Find the Taylor polynomials of degree $n=2,3,4$, centred at $x=0$, for $f(x)=\dfrac{1}{\sqrt{1+x}}$.

4. Suppose a function $f(x)$ is approximated by the 6th-degree Taylor polynomial centred at $x=0$: $P_6(x) = 3x - 4x^3 + 5x^6$.

Give the value of **(a)** $f(0)$;   **(b)** $f'(0)$;   **(c)** $f'''(0)$;   **(d)** $f^{(5)}(0)$;   **(e)** $f^{(6)}(0)$.

5. Suppose that g is a function with continuous derivatives and $g(5) = 3$, $g'(5) = -2$, $g''(5)=1$ and $g'''(5)=-3$.

   (a) What is the Taylor polynomial of degree 2 centred at $x=5$ for g?
       What is the Taylor polynomial of degree 3 centred at $x=5$ for g?

   (b) Use the two polynomials that you found in (a) to approximate $g(4.9)$.

6. Suppose $P_2(x)=a+bx+cx^2$ is the second-degree Taylor polynomial, centred at $x=0$, for a function f.

What can you say about the signs of a, b and c if f has the graph given below?
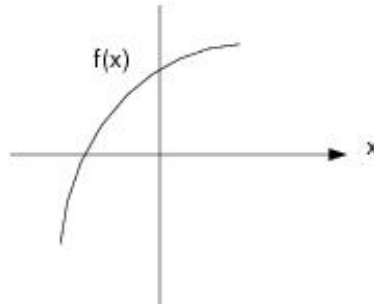
7. **(a)** Find the Taylor polynomial of degree 3 centred at $x=0$ for

$$\sinh^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1+t^2}}.$$

       Remember the Second Fundamental Theorem of Calculus?

   (b) Hence estimate $\sinh^{-1}(0.25)$.

       Check your estimate using the $\sinh^{-1}$ function in your calculator[25] and/or using numerical integration. How accurate is the estimate?

   (c) For what values of x is the approximation accurate to within 0.1?

---

[25]$\sinh^{-1}$ is in the $\boxed{\text{catalog}}$ (press S).

8. **(a)** Integrate the series for $\sin(t)/t$ (Exercise 6, Section 12.4.3) term by term, including the general term, to find the Taylor series centred at $x = 0$ for the sine-integral function
$$\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}\, dt.$$

   (b) How many terms of the series do you need to take to give an estimate for $\mathrm{Si}(0.25)$ accurate to 8 significant digits?
   Hint: Check the approximation from successive Taylor polynomials.
   Check your value using numerical integration.

9. The Taylor series of $f(x) = x^2 e^{x^2}$, centred at $x = 0$, is (Section 12.4.3)
$$x^2 + x^4 + \frac{x^6}{2!} + \frac{x^8}{3!} + \frac{x^{10}}{4!} + \cdots .$$

   Use the Taylor series to find $\left.\dfrac{d}{dx}\left(x^2 e^{x^2}\right)\right|_{x=0}$ and $\left.\dfrac{d^6}{dx^6}\left(x^2 e^{x^2}\right)\right|_{x=0}$.

10. Show how you can use the Taylor approximation for $\sin(x)/x$ centred at $x = 0$ (Exercise 6) to explain why
$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1.$$

11. By graphing f and several of its Taylor polynomials, estimate the interval of convergence of the Taylor series, centred at $x = 0$, for $f(x) = \dfrac{1}{\sqrt{1+x}}$.
   Hint: Question 3 might be useful here.

12. By looking at the Taylor series about $x = 0$ for each function, decide which of the functions
$$f(\theta) = 1 + \sin(\theta), \qquad g(\theta) = e^\theta, \qquad h(\theta) = \frac{1}{\sqrt{1-2\theta}}$$

   is largest and which is smallest for small positive θ.

   Plot the functions using a suitable $\boxed{\text{window}}$ to check your answer.

PTO

13.   *Force due to gravity*

When a body is near the surface of the Earth, we usually assume that the force due to gravity on the body is a constant $F = mg$, where m is the mass of the body and g is the acceleration due to gravity at sea level. For a body at a distance h above the surface of the Earth, a more accurate expression for the gravitational force on the body is

$$F(h) = \frac{mgR^2}{(R+h)^2},$$

where R is the radius of the Earth. Consider the situation in which the body is not too far from the surface of the Earth, so that h is much smaller than R ($h \ll R$).

(a) Express F as mg multiplied by a series in $x = h/R$.

(b) Show that F reduces to the Newtonian form $F = mg$ when $x \ll 1$.

(c) The first-order correction to the approximation $F = mg$ is obtained by taking the linear term in the series, but no higher terms. How far can you go above the Earth's surface before the first-order correction changes the approximation $F = mg$ by more than 10%? Take $R = 6400 \, \text{km}$.

Answer: 320 km.

14.   *Special Relativity*

In Einstein's Theory of Special Relativity, the mass of an object moving with velocity v is

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where $m_0 = m(0)$ is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy at velocity v and its energy at rest:

$$K(v) = m(v)c^2 - m_0 c^2.$$

(a) Use a Taylor series for K to investigate the behaviour of the kinetic energy at non-relativistic velocities. To begin, find the first three non-zero terms in the Taylor series for K in terms of $x = (v/c)^2$ about $x = 0$.

(b) Using your calculator, draw the graph of the exact $K/m_0 c^2$ against $x = (v/c)^2$, showing asymptotes.

(c) Show that when $v \ll c$, the expression for K agrees with classical Newtonian Physics: $K(v) = \frac{1}{2} m_0 v^2$.

(d) The second-order correction to the approximation $K \approx \frac{1}{2} m_0 v^2$ is obtained by taking the next term in the series, but no higher terms. Use it to estimate the change in an object's kinetic energy for $v = 3440 \, \text{m/s}$ (Mach 10). Take $c = 3 \times 10^8 \, \text{m/s}$.

Answer: $9.9 \times 10^{-9} \, \%$

PTO

15.   *Physical Chemistry*

The potential energy V of two gas molecules separated by a distance r is given by

$$V(r) = -V_0 \left(2 \left(\frac{r_0}{r} \right)^6 - \left(\frac{r_0}{r} \right)^{12} \right),$$

where V_0 and r_0 are positive constants.

(a) Show that the global minimum of V, $V = -V_0$, occurs when $r = r_0$.

(b) Sketch V/V_0 versus r/r_0 for $0.8 < r/r_0 < 1.2$.

(c) The force F between the molecules is given by $F = -dV/dr$. Derive the expression for the force and draw a graph of $F / \left(\frac{V_0}{r_0} \right)$ against r/r_0.

(d) We note that a positive F represents a repulsive force; a negative F represents an attractive force. Is F repulsive or attractive when $r = r_0$? What does this mean physically?

(e) *Qualitatively*, what happens if the molecules start with $r = r_0$ and are pulled slightly further apart?

(f) *Qualitatively*, what happens if the molecules start with $r = r_0$ and are pushed slightly closer together?

(g) The above expression for the force between the gas molecules is rather complicated and therefore not very useful. A Taylor series can help us develop a quantitative understanding of the nature of the force between the gas molecules when they are not too far from the equilibrium configuation. To do this, first write V as a Taylor series in $x = r/r_0$ centred at $x = 1$, retaining up to the quadratic term.

(h) Differentiate this result to find a series representation for the force F.

(i) By discarding all except the first non-zero term in this series, describe how the force between the atoms depends on the displacement from the equilibrium when that displacement is small.

(j) Thus, for small displacements from the equilibrium, what sort of motion results?

12.7 Solutions

12.7.1 Solutions to the exercises

Exercise 1

(a) Use the general result for Taylor polynomials to find the quadratic approximation (graph Q in the figure below) to $f(x) = e^x$ near $x = 0$.
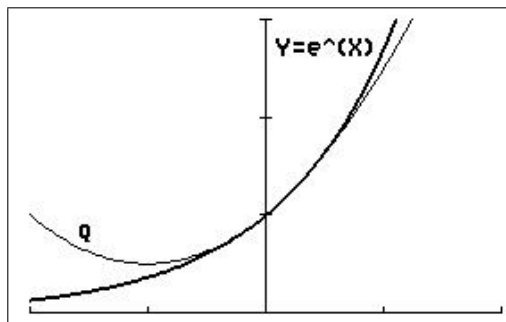
window $[-2, 2, 1] \times [-0.3, 3, 1]$

The quadratic approximation to a function f near $x = 0$ is

$$P_2(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2}\, x^2.$$

For $f(x) = e^x$, $f(0) = f'(0) = f''(0) = 1$, so that the quadratic approximation to $f(x) = e^x$ near $x = 0$ is

$$P_2(x) \;=\; 1 + x + \frac{1}{2}\, x^2.$$

(b) Use the approximation to estimate $e^{0.1}$. Compare with the calculator value and $P_1(0.1)$ (Example 1).

$e^{0.1} \approx P_2(0.1) = 1.105$

$e^{0.1} \approx P_1(0.1) = 1.1$

From the calculator, $e^{0.1} = 1.1052$ to 5 significant digits.

Exercise 2: Derive the general result for a Taylor polynomial of degree n centred at $x=0$.

Hint: Repeat and extend the process begun in Section 12.2.2, but starting with Eq. (5) for P_n instead of Eq. (1) for P_2.

Once you have established a pattern in the coefficients, you may write $\vdots$, then the nth term.

We have from Eq. (5),

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots + a_{n-1} x^{n-1} + a_n x^n. \tag{6}$$

From Eq. (6),

$$P_n(0) \;=\; a_0 \;=\; f(0).$$

$$\therefore\; a_0 \;=\; f(0).$$

Differentiating Eq. (6),

$$P_n'(x) \;=\; a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots .$$

$$\therefore\; P_n'(0) \;=\; a_1 \;=\; f'(0).$$

$$\therefore\; a_1 \;=\; f'(0).$$

$$P_n''(x) \;=\; 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots$$

$$\therefore\; P_n''(0) \;=\; 2a_2 \;=\; f''(0).$$

$$\therefore\; a_2 \;=\; \frac{f''(0)}{2} \;=\; \frac{f''(0)}{2!}.$$

$$P_n'''(x) \;=\; 6a_3 + 24a_4 x + \cdots$$

$$\therefore\; P_n'''(0) \;=\; 6a_3 \;=\; f''(0).$$

$$\therefore\; a_3 \;=\; \frac{f'''(0)}{6} \;=\; \frac{f'''(0)}{3!}.$$

$$P_n^{(4)}(x) \;=\; 24a_4 + \cdots$$

$$\therefore\; P_n^{(4)}(0) \;=\; 24a_4 \;=\; f^{(4)}(0).$$

$$\therefore\; a_4 \;=\; \frac{f^{(4)}(0)}{24} \;=\; \frac{f^{(4)}(0)}{4!}.$$

The pattern is emerging: $a_n = \dfrac{f^{(n)}(0)}{n!}.$

Therefore,

$$P_n(x) \;=\; f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Exercise 3: Find the Taylor polynomials centred at $x = 0$ for $f(x) = \sin(x)$ up to degree 11 (look for a pattern in the first few coefficients).

The general formula for the 11th-degree Taylor polynomial centred at $x = 0$ for a function f is

$$P_n(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!}\, x^2 + \frac{f'''(0)}{3!}\, x^3 + \cdots + \frac{f^{(11)}(0)}{11!}\, x^{11}.$$

$f(x) = \sin(x) \qquad f(0) = 0$

$f'(x) = \cos(x) \qquad f'(0) = 1$

$f''(x) = -\sin(x) \qquad f''(0) = 0$

$f'''(x) = -\cos(x) \qquad f'''(0) = -1$

$f^{(4)}(x) = \sin(x) \qquad f^{(4)}(0) = 0$

The pattern then repeats. Therefore, all the even powers of x have coefficient 0, all the odd powers coefficient ± 1.

Therefore, the Taylor polynomial of degree 11 centred at $x = 0$ for $f(x) = \sin(x)$ is

$$P_{11}(x) \;=\; x \;-\; \frac{x^3}{3!} \;+\; \frac{x^5}{5!} \;-\; \frac{x^7}{7!} \;+\; \frac{x^9}{9!} \;-\; \frac{x^{11}}{11!}.$$

This contains all the other Taylor polynomials up to degree 11.

Plot P_1 and P_3 on the relevant graphs.
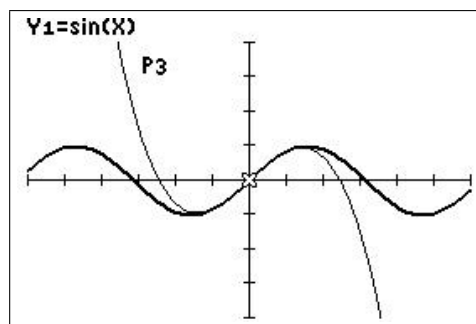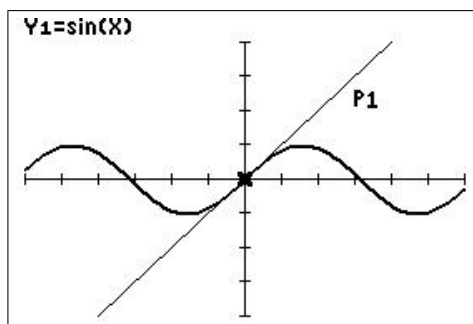
window $[-6, 6, 1] \times [-4, 4, 1]$

Use each polynomial to approximate $\sin(0.2)$, and compare with the calculator value. Underline the digits that are correct in each approximate value.

$P_0(x) = 0$ for all x. P_n for even n is the same as P_{n-1}.

From the calculator, $\sin(0.2) = 0.19866933080494$ to 14 significant digits.

$\sin(0.2) \approx P_1(0.2) = 0.2$

$\sin(0.2) \approx P_3(0.2) = \underline{0.198666}\dot{6}$

$\sin(0.2) \approx P_5(0.2) = \underline{0.19866933}\dot{3}$

$\sin(0.2) \approx P_7(0.2) = \underline{0.1986693308}$

$\sin(0.2) \approx P_9(0.2) = \underline{0.19866933080}635$

$\sin(0.2) \approx P_{11}(0.2) = \underline{0.19866933080494}$

Exercise 4: Find the fifth-degree Taylor polynomial for $f(x) = \dfrac{1}{x}$ centred at $x = 3$.

$$P_5(x) = f(3) + f'(3)\,(x-3) + \frac{f''(3)}{2!}\,(x-3)^2 + \frac{f'''(3)}{3!}\,(x-3)^3 + \frac{f^{(4)}(3)}{4!}\,(x-3)^4 - \frac{f^{(5)}(3)}{5!}\,(x-3)^5.$$

$$f(x) = \frac{1}{x} \qquad f(3) = \frac{1}{3}$$

$$f'(x) = -\frac{1}{x^2} \qquad f'(3) = -\frac{1}{3^2}$$

$$f''(x) = \frac{2}{x^3} \qquad f''(3) = \frac{2}{3^3} = \frac{2!}{3^3}$$

$$f'''(x) = -\frac{6}{x^4} \qquad f'''(3) = -\frac{6}{3^4} = -\frac{3!}{3^4}$$

$$f^{(4)}(x) = \frac{24}{x^5} \qquad f^{(4)}(3) = \frac{24}{3^5} = \frac{4!}{3^5}$$

$$f^{(5)}(x) = -\frac{120}{x^6} \qquad f^{(5)}(3) = -\frac{120}{3^6} = -\frac{5!}{3^6}$$

$$\therefore \; P_5(x) \;=\; \frac{1}{3} - \frac{x-3}{3^2} + \frac{(x-3)^2}{3^3} - \frac{(x-3)^3}{3^4} + \frac{(x-3)^4}{3^5} - \frac{(x-3)^5}{3^6}.$$

Plot f and $P_1 - P_5$ on your calculator using a window $[-1, 9, 1] \times [-2, 2, 1]$.
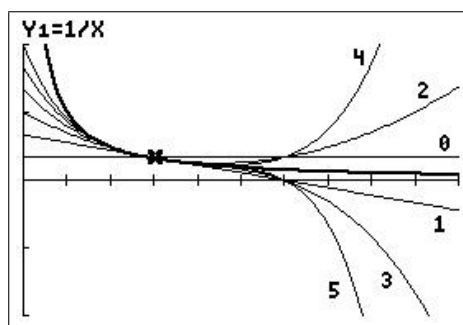
Exercise 5: Find the Taylor series centred at $x = \pi/2$ for $f(x) = \cos(x)$.

$$f(x) = f(\pi/2) + f'(\pi/2)\,(x - \pi/2) + \frac{f''(\pi/2)}{2!}\,(x - \pi/2)^2 + \frac{f'''(\pi/2)}{3!}\,(x - \pi/2)^3 + \cdots .$$

$f(x) = \cos(x) \qquad f(\pi/2) = 0$

$f'(x) = -\sin(x) \qquad f'(\pi/2) = -1$

$f''(x) = -\cos(x) \qquad f''(\pi/2) = 0$

$f'''(x) = \sin(x) \qquad f'''(\pi/2) = 1$

$f^{(4)}(x) = \cos(x) \qquad f^{(4)}(\pi/2) = 0$

The pattern then repeats. Therefore, all the even powers of $x - \pi/2$ have coefficient 0, all the odd powers coefficient ± 1.

Therefore, the Taylor series centred at $x = \pi/2$ for $f(x) = \cos(x)$ is

$$\cos(x) \;=\; -(x - \pi/2) \;+\; \frac{(x - \pi/2)^3}{3!} \;-\; \frac{(x - \pi/2)^5}{5!} \;+\; \frac{(x - \pi/2)^7}{7!} \;-\; \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{(x - \pi/2)^{2n-1}}{(2n-1)!}.$$
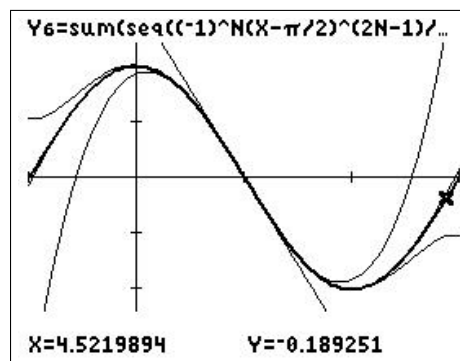
window $[-\pi/2, 3\pi/2, \pi/2] \times [-1.2, 1.2, 0.5]$

The cursor on the plot is on P_{21} (Y6). $P_{21}(x) = \cos(x)$ at this point to 9 decimal places.

Exercise 6: Find the Taylor series for $f(t) = \dfrac{\sin(t)}{t}$ centred at $t = 0$.

From the box on page 94,

$$\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + \cdots .$$

$$\therefore \quad \frac{\sin(t)}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots + (-1)^n \frac{t^{2n}}{(2n+1)!} + \cdots .$$
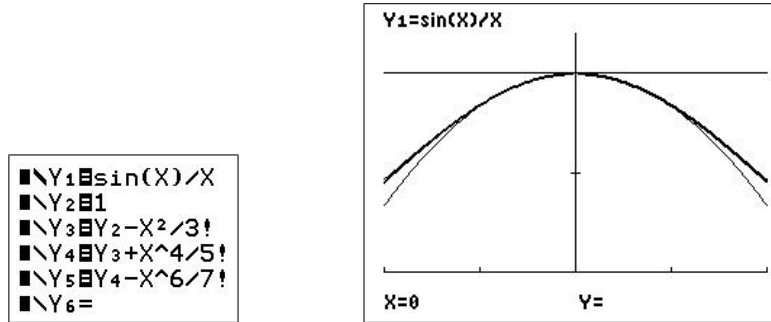
window $[-2, 2, 1] \times [0, 1.2, 0.5]$

Exercise 7: Use the standard method to find the first three non-zero terms in the Taylor series centred at $x = 0$ for $f(x) = e^{x^2}$.

$$P_4(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!}\, x^2 + \frac{f'''(0)}{3!}\, x^3 + \frac{f^{(4)}(0)}{4!}\, x^4 + \cdots .$$

$f(x) = e^{x^2} \qquad f(0) = 1$

$f'(x) = 2x e^{x^2} \qquad f'(0) = 0$

$f''(x) = 2e^{x^2} + 4x^2 e^{x^2} \qquad f''(0) = 2$

$f'''(x) = 8x e^{x^2} + (2 + 4x^2) \cdot 2x e^{x^2} \qquad f'''(0) = 0$

$f^{(4)}(x) = (12 + 24x^2) e^{x^2} + (10x + 8x^3) \cdot 2x e^{x^2} \qquad f^{(4)}(0) = 12$

$$\therefore \quad e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \cdots .$$

Note that $f(x) = e^{x^2}$ and $P_4(x)$ are even functions: Taylor series centred at $x = 0$ of even (odd) functions only contain even (odd) powers of x. Plot together with f.
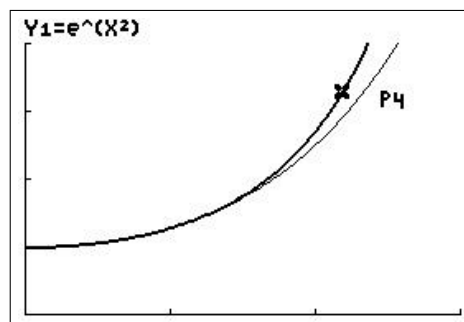
window $[0, 1.5, 0.5] \times [0, 4, 1]$

Exercise 8: From the Taylor polynomials for $f(x) = \ln(x)$ centred at $x = 1$ in Example 3, write down the Taylor series and determine its interval of convergence by graphing the function and its first few Taylor polynomials.

$$\ln(x) \;=\; x-1-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\frac{(x-1)^5}{5}+\cdots$$

$$\;=\; \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}.$$
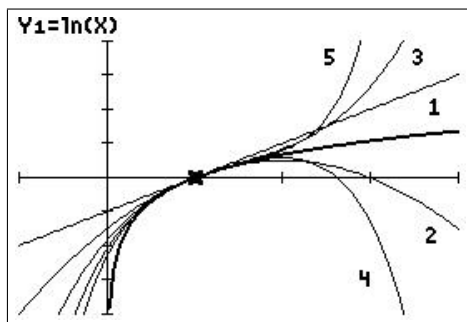
window $[-1, 4, 1] \times [-4, 4, 1]$

The interval of convergence, from the graph, is $0 < x < 2$.

Check by evaluating $P_n(1.2)$ and $P_n(2.2)$ for the first few values of n and comparing them with the values of $\ln(1.2)$ and $\ln(2.2)$.

$\ln(1.2) = 0.1823216$ and $\ln(2.2) = 0.7884574$

n	$P_n(1.2)$	$P_n(2.2)$
1	0.2	1.2
2	0.18	0.48
3	$0.182\dot{6}$	1.056
4	$0.1822\dot{6}$	0.5376
5	$0.182330\dot{6}$	1.035264
	converges	*diverges*

Exercise 9: Find the Taylor series centred at $x=0$ for $f(x)=\dfrac{1}{1-x}$ and determine its interval of convergence.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots .$$

$f(x) = \dfrac{1}{1-x} \qquad f(0) = 1$

$f'(x) = \dfrac{1}{(1-x)^2} \qquad f'(0) = 1$

$f''(x) = \dfrac{2}{(1-x)^3} \qquad f''(0) = 2 = 2!$

$f'''(x) = \dfrac{6}{(1-x)^4} \qquad f'''(0) = 6 = 3!$

$f^{(4)}(x) = \dfrac{24}{(1-x)^5} \qquad f^{(4)}(0) = 24 = 4!$

$f^{(5)}(x) = \dfrac{120}{(1-x)^6} \qquad f^{(5)}(0) = 120 = 5!$

$$\therefore \;\; f(x) \;\; = \;\; 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \;\; = \;\; \sum_{n=0}^{\infty} x^n .$$

```
■\Y₁☐1/(1-X)
■\Y₂☐1
■\Y₃☐Y₂+X
■\Y₄☐Y₃+X²
■\Y₅☐Y₄+X³
■\Y₆☐Y₅+X^4
■\Y₇☐Y₆+X^5
■\Y₈☐sum(seq(X^N,N,0,20,1)
```
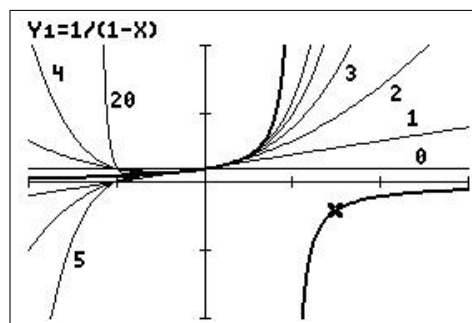
window $[-2, 3, 1] \times [-10, 10, 5]$

$f(x) = 1/(1-x)$ has an infinite discontinuity at $x = 1$. The graphs of the first few Taylor polynomials show that the fit to f is good for $-1 < x < 1$, and improves as the degree of the polynomial increases. However, the polynomials clearly diverge from f outside that interval, particularly obvious for the second branch of f at $x > 1$. This is the interval of convergence.

Exercise 10: Find the Taylor series for the function

$$f(x) = \begin{cases} \cos(x) & |x| \leqslant \pi \\ -1 & |x| > \pi \end{cases}$$

centred at $x=0$.

The Taylor series for f is the same as the Taylor series for $\cos(x)$, because the series is centred on $x=0$, at which $f(x)=\cos(x)$.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

What is its interval of convergence?

The interval of convergence for the function f here is $-\pi \leqslant x \leqslant \pi$, whereas the series for $\cos(x)$ converges for all x.
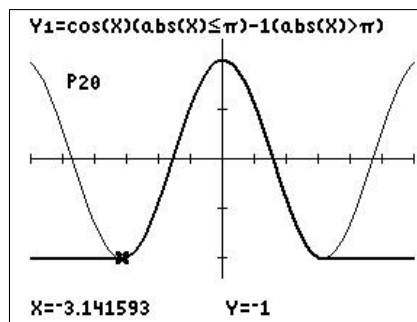
window $[-6, 6, 1] \times [-1.2, 1.2, 0.5]$

PTO

Exercise 11

(a) Find the Taylor series and its interval of convergence for $\ln(1+x)$ about $x=0$.

$$f(x) = \ln(1+x) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \qquad f(0) = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \qquad f''(0) = 2 = 2!$$

$$f^{(4)}(x) = -\frac{6}{(1+x)^4} \qquad f'''(0) = -6 = -3!$$

$$f^{(5)}(x) = \frac{24}{(1+x)^5} \qquad f^{(4)}(0) = 24 = 4!$$

Therefore,

$$
\begin{aligned}
\ln(1+x) &= f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \frac{f^{(4)}(0)}{4!}\,x^4 + \frac{f^{(5)}(0)}{5!}\,x^5 + \dots \\[2mm]
&= x - \frac{1}{2!}\,x^2 + \frac{2!}{3!}\,x^3 - \frac{3!}{4!}\,x^4 + \frac{4!}{5!}\,x^5 + \dots \\[2mm]
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots \\[2mm]
&= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}.
\end{aligned}
$$

The graphs of $f(x) = \ln(1+x)$ and $P_1 - P_5$ are plotted in the figure below.
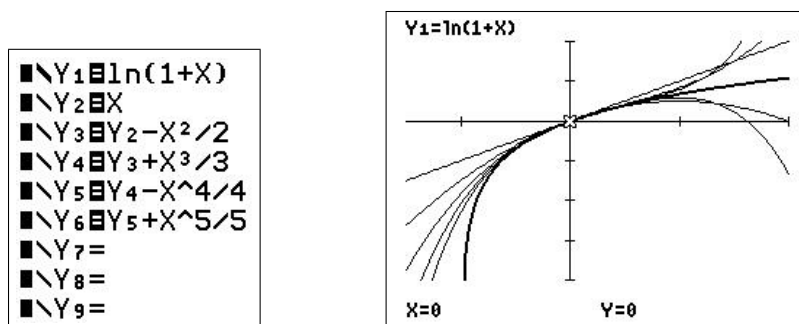
window $[-1.5, 2, 1] \times [-4, 2, 1]$

The Taylor series for $\ln(1+x)$ about $x=0$ is therefore $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$.

From the graph, the interval of convergence is $-1 < x < 1$.

(b) The Taylor series for $\ln(x)$ about $x=1$ has an IC of $0<x<2$ (Example 3) and, as you found in (a), the series for $f(x)$ about $x=0$ also has a very restricted IC. Neither is therefore suitable for finding $\ln(x)$ over its whole domain $x>0$.

Find the Taylor series for the function

$$f(x) = \ln\left(\frac{1+x}{1-x}\right)$$

and its interval of convergence. *Hint:* use a log law and previous results.

From (a),

$$
\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots \qquad -1<x<1 \\[2mm]
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad -1<x<1.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \ \ln(1-x) &= (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} + \cdots \\[2mm]
&= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots \qquad -1<x<1 \\[2mm]
&= -\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad -1<x<1.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \ \ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \qquad \text{log law} \\[2mm]
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots \qquad -1<x<1 \\[2mm]
&\quad - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots\right) \qquad -1<x<1 \\[2mm]
&= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots \qquad -1<x<1 \\[2mm]
&= 2\sum_{n=1,3,5,\ldots}^{\infty} \frac{x^n}{n} \qquad -1<x<1 \\[2mm]
&= 2\sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1} \qquad -1<x<1 \qquad (n=2k-1).
\end{aligned}
$$

(c) As x ranges across the interval of convergence you found in (a), what range of
values does $g(x) = \dfrac{1+x}{1-x}$ take?
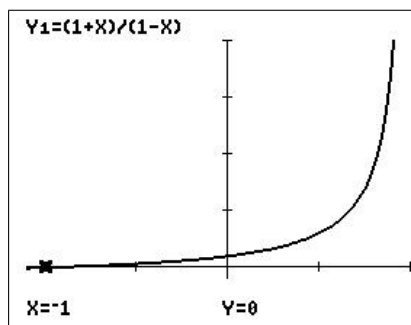
window $[-1.1, 1, 0.5] \times [-1, 20, 5]$

The graph $y = \dfrac{1+x}{1-x}$ above shows that for $-1 < x < 1$, $0 < g(x) = \dfrac{1+x}{1-x} < \infty$, that
is g takes all values greater than 0.

Hence, given some number $a > 0$, we can find an x, $-1 < x < 1$, such that $g(x) = a$. Then,
using the series above we can find $\ln(a)$ for all $a > 0$.

(d) Hence write out an algorithm to find the natural log of any number $x > 0$.

If $\dfrac{1+x}{1-x} = a$, then $x = \dfrac{a-1}{a+1}$. Hence, from our series above,

$$\ln(a) \;=\; 2 \sum_{k=1}^{\infty} \frac{\left(\dfrac{a-1}{a+1}\right)^{2k-1}}{2k-1} \qquad a > 0.$$

Replacing a by a general variable x in the series above, we have our final result, a series
for the natural log of all positive numbers,

$$\ln(x) \;=\; 2 \sum_{k=1}^{\infty} \frac{\left(\dfrac{x-1}{x+1}\right)^{2k-1}}{2k-1} \qquad x > 0.$$

Plot the difference between $\ln(x)$ and $P_5(x)$, and the difference between $\ln(x)$ and $P_{10}(x)$,
for $0 < x < 20$.
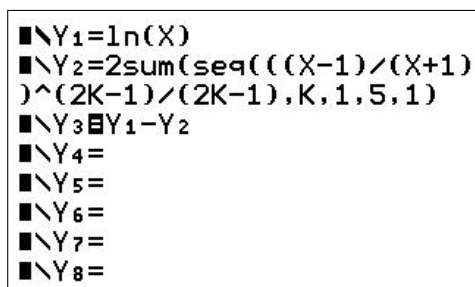
Set up for $\ln(x) - P_5(x)$
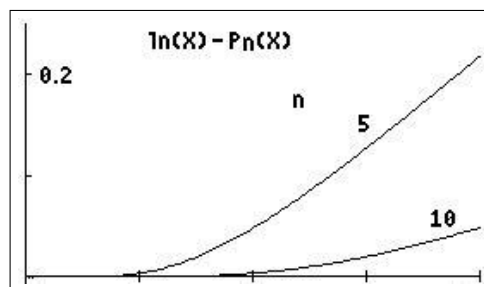
window $[0, 20, 5] \times [0, 0.25, 0.1]$

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12.7.2   Solutions to the problems

1. Find the Taylor polynomials of degree $n=2,3,4$, centred at $x=0$, for $f(x)=\sqrt{1+x}$.

$$f(x) = \sqrt{1+x} = (1+x)^{1/2} \qquad f(0) = 1$$

$$f'(x) = \frac{1}{2(1+x)^{1/2}} \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4(1+x)^{3/2}} \qquad f''(0) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8(1+x)^{5/2}} \qquad f'''(0) = \frac{3}{8}$$

$$f^{(4)}(x) = -\frac{15}{16(1+x)^{7/2}} \qquad f^{(4)}(0) = -\frac{15}{16}$$

Therefore, using the general formula,

$$
\begin{aligned}
P_2(x) &= f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 \\[2mm]
&= 1 + \frac{1}{2}\,x - \frac{1}{4}\frac{x^2}{2!} \\[2mm]
&= 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2.
\end{aligned}
$$

$$
\begin{aligned}
P_3(x) &= P_2(x) + \frac{f'''(0)}{3!}\,x^3 \\[2mm]
&= P_2(x) + \frac{3}{8}\frac{x^3}{3!} \\[2mm]
&= 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2 + \frac{1}{16}\,x^3.
\end{aligned}
$$

$$
\begin{aligned}
P_4(x) &= P_3(x) + \frac{f^{(4)}(0)}{4!}\,x^4 \\[2mm]
&= P_3(x) - \frac{15}{16}\frac{x^4}{4!} \\[2mm]
&= 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2 + \frac{1}{16}\,x^3 - \frac{5}{128}\,x^4.
\end{aligned}
$$


```
Y1=√(1+X)

■\Y₁目√(1+X)
■\Y₂目1
■\Y₃目Y₂+X∕2
■\Y₄目Y₃−X²∕8
■\Y₅目Y₄+X³∕16
■\Y₆目Y₅−5X^4∕128
■\Y₇=
■\Y₈=
■\Y₉=
```

window $[-1, 2, 1] \times [0, 2, 1]$

2. Find the Taylor polynomials of degree $n=3,4$, centred at $x=0$, for
$f(x) = \arctan(x) \equiv \tan^{-1}(x)$.

$f(x) = \arctan(x) \qquad f(0) = 0$

$f'(x) = \dfrac{1}{1+x^2} \qquad f'(0) = 1$

$f''(x) = -\dfrac{2x}{(1+x^2)^2} \qquad f''(0) = 0$

$f'''(x) = -\dfrac{2}{(1+x^2)^2} + \dfrac{8x^2}{(1+x^2)^3} \qquad f'''(0) = -2$

$f^{(4)}(x) = \dfrac{8x}{(1+x^2)^3} + \dfrac{16x}{(1+x^2)^3} - \dfrac{48x^3}{(1+x^2)^4} \qquad f^{(4)}(0) = 0$

Therefore, using the general formula,

$$
\begin{aligned}
P_3(x) = P_4(x) \;&=\; f(0) \;+\; f'(0)\,x \;+\; \frac{f''(0)}{2!}\,x^2 \;+\; \frac{f'''(0)}{3!}\,x^3 \\[2mm]
&=\; x \;-\; 2\,\frac{x^3}{3!} \\[2mm]
&=\; x \;-\; \frac{1}{3}\,x^3.
\end{aligned}
$$


```
■\Y1日tan⁻¹(X)
■\Y2日X-X³/3
■\Y3=
```
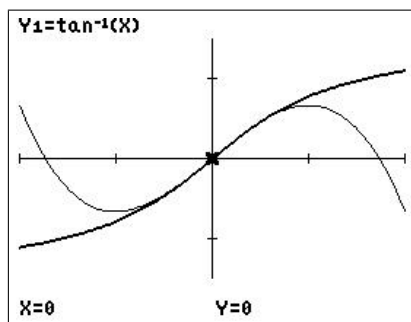
window $[-2, 2, 1] \times [-1.5, 1.5, 1]$

3. Find the Taylor polynomials of degree $n = 2, 3, 4$, centred at $x = 0$, for
$$f(x) = \frac{1}{\sqrt{1+x}}.$$

$$f(x) = \frac{1}{\sqrt{1+x}} \equiv \frac{1}{(1+x)^{1/2}} \equiv (1+x)^{-1/2} \qquad f(0) = 1$$

$$f'(x) = -\frac{1}{2(1+x)^{3/2}} \qquad f'(0) = -\frac{1}{2}$$

$$f''(x) = \frac{3}{4(1+x)^{5/2}} \qquad f''(0) = \frac{3}{4}$$

$$f'''(x) = -\frac{15}{8(1+x)^{7/2}} \qquad f'''(0) = -\frac{15}{8}$$

$$f^{(4)}(x) = \frac{105}{16(1+x)^{9/2}} \qquad f^{(4)}(0) = \frac{105}{16}$$

Therefore, using the general formula,

$$\begin{aligned}
P_2(x) &= f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 \\
&= 1 - \frac{1}{2}\,x + \frac{3}{4 \times 2!}\,x^2 \\
&= 1 - \frac{1}{2}\,x + \frac{3}{8}\,x^2.
\end{aligned}$$

$$\begin{aligned}
P_3(x) &= P_2(x) + \frac{f'''(0)}{3!}\,x^3 \\
&= P_2(x) - \frac{15}{8 \times 3!}\,x^3 \\
&= 1 - \frac{1}{2}\,x + \frac{3}{8}\,x^2 - \frac{5}{16}\,x^3.
\end{aligned}$$

$$\begin{aligned}
P_4(x) &= P_3(x) + \frac{f^{(4)}(0)}{4!}\,x^4 \\
&= P_3(x) + \frac{105}{16 \times 4!}\,x^4 \\
&= 1 - \frac{1}{2}\,x + \frac{3}{8}\,x^2 - \frac{5}{16}\,x^3 + \frac{35}{128}\,x^4.
\end{aligned}$$

```
■\Y₁⊟1/√(1+X)
■\Y₂⊟1
■\Y₃⊟Y₂-X/2
■\Y₄⊟Y₃+3X²/8
■\Y₅⊟Y₄-5X³/16
■\Y₆⊟Y₅+35X^4/128
■\Y₇=
■\Y₈=
■\Y₉=
```
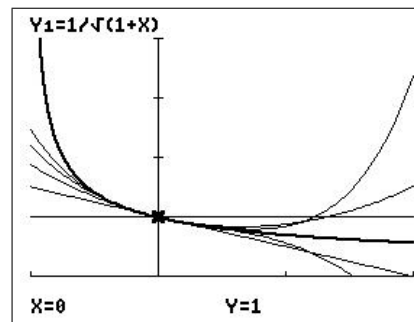

window $[-1, 2, 1] \times [0, 4, 1]$

4. Suppose a function $f(x)$ is approximated by the sixth-degree Taylor polynomial centred at $x=0$

$$P_6(x) = 3x - 4x^3 + 5x^6. \tag{7}$$

Give the value of **(a)** $f(0)$;   **(b)** $f'(0)$;   **(c)** $f'''(0)$;   **(d)** $f^{(5)}(0)$;   **(e)** $f^{(6)}(0)$.

The general formula for P_6 is

$$P_6(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \frac{f^{(4)}(0)}{4!}\,x^4 + \frac{f^{(5)}(0)}{5!}\,x^5 + \frac{f^{(6)}(0)}{6!}\,x^6. \tag{8}$$

(a) Setting $x=0$ in Eqs. (7) and (8) gives $P_6(0)=f(0)=0$.

(b) Differentiating Eq. (7) and setting $x=0$ gives $P_6'(0)=3$.
Differentiating Eq. (8) and setting $x=0$ gives $P_6'(0)=f'(0)$.
Therefore, $f'(0)=3$.

(c) Differentiating Eq. (7) twice more and setting $x=0$ gives $P_6'''(0)=-24$.
Differentiating Eq. (8) twice more and setting $x=0$ gives
$P_6'''(0)=f'''(0)$.
Therefore, $f'''(0)=-24$.

Continuing this process, we have

(d) $f^{(5)}(0)=P_6^{(5)}(0)=0$, and

(e) $f^{(6)}(0)=P_6^{(5)}(0)=3600$.

5. Suppose g is a function which has continuous derivatives, and that $g(5)=3$, $g'(5)=-2$, $g''(5)=1$, $g'''(5)=-3$.

(a) What is the Taylor polynomial of degree 2 centred at $x=5$ for g?
What is the Taylor polynomial of degree 3 centred at $x=5$ for g?

$$
\begin{aligned}
P_2(x) &= g(5) + g'(5)(x-5) + \frac{g''(5)}{2!}(x-5)^2 \\[2mm]
&= 3 - 2(x-5) + \frac{1}{2}(x-5)^2.
\end{aligned}
$$

$$
\begin{aligned}
P_3(x) &= P_2(x) + \frac{g'''(5)}{3!}(x-5)^3 \\[2mm]
&= 3 - 2(x-5) + \frac{1}{2}(x-5)^2 - \frac{1}{2}(x-5)^3.
\end{aligned}
$$

(b) Use the two polynomials that you found in (a) to approximate $g(4.9)$.

$g(4.9) \approx P_2(4.9) = 3.205.$
$g(4.9) \approx P_3(4.9) = 3.2055.$

6. Suppose $P_2(x) = a + bx + x^2$ is the second-degree Taylor polynomial centred at $x = 0$ for a function f.

What can you say about the signs of a, b and c if f has the graph given below?
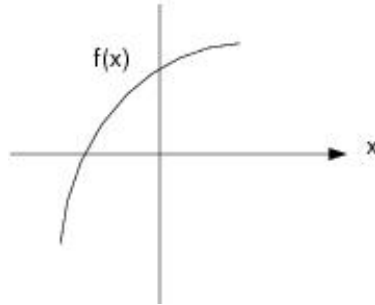
We know $f(0) = P_2(0) = a$. From the graph, $f(0) > 0$, so that a is positive.

Similarly, $f'(0) = P_2'(0) = b$. The slope of the graph at $x = 0$ is positive, so that b is positive.

$f''(0) = P_2''(0) = 2c$. The graph of f is concave down at $x = 0$, so that f'' is negative and therefore c is negative.

7. (a) Find the Taylor polynomial of degree 3 centred at $x = 0$ for

$$\sinh^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1+t^2}}$$

$$f(x) = \int_0^x \frac{dt}{\sqrt{1+t^2}} \qquad f(0) = 0$$

$$f'(x) = \frac{1}{\sqrt{1+x^2}} \qquad f'(0) = 1 \quad \text{Second Fundamental Theorem of Calculus}$$

$$f''(x) = -\frac{x}{(1+x^2)^{3/2}} \qquad f''(0) = 0$$

$$f'''(x) = \frac{2x^2 - 1}{(1+x^2)^{5/2}} \qquad f'''(0) = -1$$

$$\therefore \quad P_3(x) \quad = \quad f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= \quad 0 + x + 0 - \frac{x^3}{3!}$$

$$= \quad x - \frac{x^3}{3!}.$$

Y1=sinh⁻¹(X)

■\Y₁⊟sinh⁻¹(X)
■\Y₂⊟X−X³/3!

X=0   Y=0

window $[-2, 2, 1] \times [-2, 2, 1]$
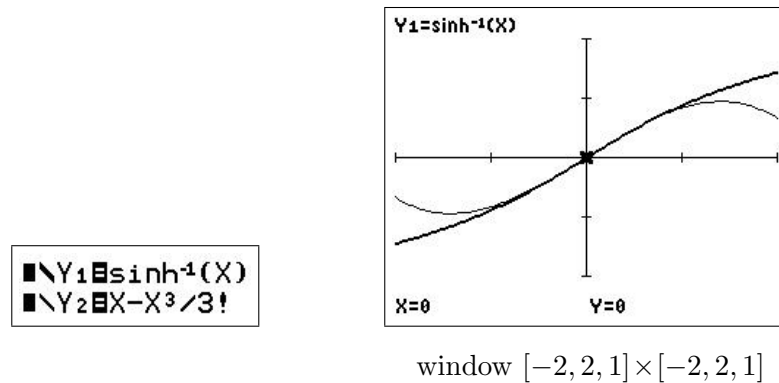
(b) Hence estimate $\sinh^{-1}(0.25)$. Check your estimate using the $\sinh^{-1}$ function in your calculator and/or using numerical integration. How accurate is the estimate?

$P_3(0.25) = 0.247396$ (6 SD).

$\sinh^{-1}(0.25) = 0.247466$ (6 SD), using either the calculator $\sinh^{-1}$ function or numerical integration.

The error in P_3 is 7×10^{-5}.

(c) For what values of x is the approximation accurate to within 0.1?

Plot the graph of $\left| \sinh^{-1}(x) - P_3(x) \right|$ and use *intersect* to find where this function is equal to 0.1.

Y1=abs(sinh⁻¹(X)−X+X³/6)

Intersection
X=1.1888163   Y=0.1
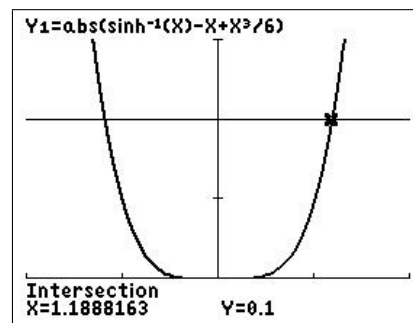
When $-1.189 < x < 1.189$ (4 SD), the error in P_3 is less than 0.1.

8. (a) Integrate the series for $\sin(t)/t$ (Exercise 5) term by term, including the general term, to find the Taylor series centred at $x = 0$ for the sine-integral function

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t} \, dt.$$

$$
\begin{aligned}
\text{Si}(x) &= \int_0^x \frac{\sin(t)}{t} \, dt \\
&= \int_0^x \left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots + (-1)^n \frac{t^{2n}}{(2n+1)!} + \cdots \right) dt \\
&= x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1) \cdot (2n+1)!} + \cdots .
\end{aligned}
$$

(b) How many terms of the series do you need to take to give an estimate for Si(0.25) accurate to 8 significant digits?

Evaluating the successive Taylor-polynomial approximations, $P_n(0.25)$, to Si(0.25), we obtain

n	$P_n(0.25)$
1	0.25000000
3	0.24913194
5	0.24913357
7	0.24913357

Therefore, because $P_5(0.25) = P_7(0.25)$ to 8 significant digits, $P_5(0.25) = 0.24913357$ is accurate to 8 significant digits.

Check your value using numerical integration.

fnInt gives Si(0.25) $= 0.2491335703$ (all displayed digits), confirming the answer above.

9. The Taylor series centred at $x = 0$ for $f(x) = x^2 e^{x^2}$ is

$$x^2 \;+\; x^4 \;+\; \frac{x^6}{2!} \;+\; \frac{x^8}{3!} \;+\; \frac{x^{10}}{4!} + \cdots .$$

Use the Taylor series to find $\left. \dfrac{d}{dx}\left(x^2 e^{x^2}\right)\right|_{x=0}$.

Differentiating the series term by term, we have

$$\frac{d}{dx}\left(x^2 e^{x^2}\right) = 2x \;+\; 4x^3 \;+\; \cdots ,$$

giving

$$\left. \frac{d}{dx}\left(x^2 e^{x^2}\right)\right|_{x=0} = 0.$$

Find $\left. \dfrac{d^6}{dx^6}\left(x^2 e^{x^2}\right)\right|_{x=0}$.

Differentiating the series term by term to find the sixth derivative, we have

$$\frac{d^6}{dx^6}\left(x^2 e^{x^2}\right) = \frac{6!}{2!} \;+\; \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{3!}x^2 \;+\; \cdots ,$$

giving

$$\left. \frac{d^6}{dx^6}\left(x^2 e^{x^2}\right)\right|_{x=0} = \frac{6!}{2!} = 360.$$

10. Show how you can use the Taylor approximation for $\sin(x)/x$ centred at $x=0$ (Exercise 6) to explain why

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

From Exercise 6, for x near 0,

$$\frac{\sin(x)}{x} \approx 1 - \frac{x^3}{3!}.$$

Letting $x \to 0$, we obtain the result.

11. By graphing f and several of its Taylor polynomials, estimate the interval of convergence of the Taylor series centred at $x=0$ for $f(x) = \dfrac{1}{\sqrt{1+x}}$.

From Question 3,

$$P_4(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4.$$
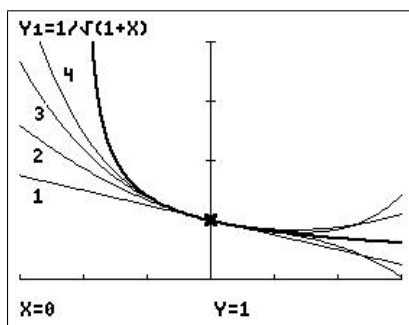
window $[-1.5, 1.5, 0.5] \times [-0.4, 4, 1]$

The divergence of the polynomials for x greater than about 1 is reasonably obvious, but it is not so clear at $x=-1$. However, the interval of convergence is always symmetrical about the point at which the polynomials are centred ($x=0$ here).

We could also argue in this case that, as f is undefined for $x \leqslant -1$, any polynomial (all of which are defined for $x \leqslant -1$) cannot converge to f for $x \leqslant -1$.

The interval of convergence is $-1 < x < 1$.

12. By looking at the Taylor series about $x = 0$ for each function, decide which of the functions
$$f(\theta) = 1 + \sin(\theta), \qquad g(\theta) = e^{\theta}, \qquad h(\theta) = \frac{1}{\sqrt{1-2\theta}}$$
is largest and which is smallest for small positive θ.

Using the given series for $\sin(\theta)$ and e^{θ}, and the standard Taylor-series method for $\dfrac{1}{\sqrt{1-2\theta}}$:

$$f(\theta) \;=\; 1 + \theta - \frac{\theta^3}{6} + \cdots .$$

$$g(\theta) \;=\; 1 + \theta + \frac{\theta^2}{2} + \frac{\theta^3}{6} + \cdots$$

$$h(\theta) \;=\; (1-2\theta)^{-1/2}.$$

$$\;=\; 1 + \theta + \frac{3\theta^2}{2} + \cdots .$$

Comparing the third term in each series, we see that $f(\theta) < g(\theta) < h(\theta)$ for small positive θ.

Plot the functions using a suitable $\boxed{\text{window}}$ to check your answer.
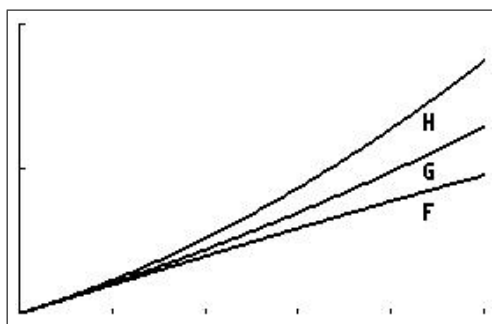
window $[0, 0.5, 0.1] \times [1, 2, 0.5]$

The plot verifies the algebraic result.

13.   *Force due to gravity*

When a body is near the surface of the Earth, we usually assume that the force due to gravity on the body is a constant mg, where m is the mass of the body and g is the acceleration due to gravity at sea level. For a body at a distance h above the surface of the Earth, a more accurate expression for the gravitational force on the body is

$$F(h) = \frac{mgR^2}{(R+h)^2},$$

where R is the radius of the Earth. Consider the situation in which the body is not too far from the surface of the Earth, so that h is much smaller than R ($h \ll R$).

(a) Express F as mg multiplied by a series in $x = h/R$.

Let $x = h/R$. Then,

$$
\begin{aligned}
F(h) &= \frac{mgR^2}{(R+h)^2} \\[2mm]
&= mg\,\frac{R^2}{(R+h)^2} \\[2mm]
&= mg\,\frac{R^2}{R^2\left(1+\dfrac{h}{R}\right)^2} \\[2mm]
&= mg\,\frac{1}{\left(1+\dfrac{h}{R}\right)^2}. \\[2mm]
\therefore\ F(x) &= mg\,\frac{1}{(1+x)^2} \qquad x = \frac{h}{R} \\[2mm]
&= mg\,(1+x)^{-2}.
\end{aligned}
$$

Find the Taylor series for $g(x) = (1+x)^{-2}$ about $x = 0$.

$g(x) = (1+x)^{-2} \qquad g(0) = 1$

$g'(x) = -2(1+x)^{-3} \qquad g'(0) = -2$

$g''(x) = 6(1+x)^{-4} \qquad g''(0) = 6$

Then $(1+x)^{-2} = 1 - 2x + \dfrac{6x^2}{2!} + \cdots = 1 - 2x + 3x^2 + \ldots$

Therefore,

$$F = mg\left(1 - 2x + 3x^2 + \cdots\right) = mg\left(1 - \frac{2h}{R} + 3\left(\frac{h}{R}\right)^2 + \cdots\right).$$

(b) Show that F reduces to the Newtonian form $F=mg$ when $x\ll 1$.

When $x=h/R\ll 1$, we can neglect all but the first term of the series above. This gives directly $F=mg$.

(c) The first-order correction to the approximation $F=mg$ is obtained by taking the linear term in the series, but no higher terms. How far can you go above the Earth's surface before the first-order correction changes the approximation $F=mg$ by more than 10%? Take $R=6400\,\text{km}$.

The ratio of the first-order term to the zeroth-order term is $\dfrac{2h/R}{1}=\dfrac{2h}{R}$.

For this correction to be 10%, we require $\dfrac{2h}{R}=0.1$, giving $h=0.05R=320$.

Therefore, we can go up to $320\,\text{km}$ above the Earth's surface before the first-order correction is 10%.

PTO

14.   *Special Relativity*

In Einstein's Theory of Special Relativity, the mass of an object moving with velocity v is

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where $m_0 = m(0)$ is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy at velocity v and its energy at rest:

$$K(v) = m(v)c^2 - m_0 c^2.$$

(a) Use a Taylor series for K to investigate the behaviour of the kinetic energy at non-relativistic velocities. To begin, find the first three non-zero terms in the Taylor series for K in terms of $x = (v/c)^2$ about $x = 0$.

$$
\begin{aligned}
K(m) &= mc^2 - m_0 c^2 \\[2mm]
&= \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2 \\[2mm]
&= m_0 c^2 \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1 \right). \\[2mm]
\therefore\ K(x) &= m_0 c^2 \left((1-x)^{-1/2} - 1 \right) \qquad x = (v/c)^2.
\end{aligned}
$$

Find the Taylor series for $g(x) = (1-x)^{-1/2} - 1$ about $x = 0$.

$$g(x) = (1-x)^{-1/2} - 1 \qquad g(0) = 0$$

$$g'(x) = \frac{(1-x)^{-3/2}}{2} \qquad g'(0) = \frac{1}{2}$$

$$g''(x) = \frac{3(1-x)^{-5/2}}{4} \qquad g''(0) = \frac{3}{4}$$

$$g'''(x) = \frac{15(1-x)^{-7/2}}{8} \qquad g'''(0) = \frac{15}{8}$$

Therefore,

$$g(x) = \frac{1}{2}x + \frac{3}{4}\frac{x^2}{2!} + \frac{15}{8}\frac{x^3}{3!} + \cdots = \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots .$$

Therefore,

$$K(x) = m_0 c^2 \left(\frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots \right).$$

$$\therefore\ K(v) = m_0 c^2 \left(\frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \frac{5}{16}\left(\frac{v}{c}\right)^6 + \cdots \right).$$

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(b) Using your calculator, draw the graph of the exact K/m_0c^2 against $x = (v/c)^2$, showing asymptotes.

$$K(v) = mc^2 - m_0c^2.$$

$$\therefore \frac{K}{m_0c^2} = \frac{m}{m_0} - 1$$

$$= \frac{1}{\sqrt{1-v^2/c^2}} - 1$$

$$= \frac{1}{\sqrt{1-x}} - 1.$$

Graphing $K(x)/m_0c^2$ against x, we get


```
Y1=1/√(1-X)-1



                              ✖

X=0.8484848        Y=1.5690465
```
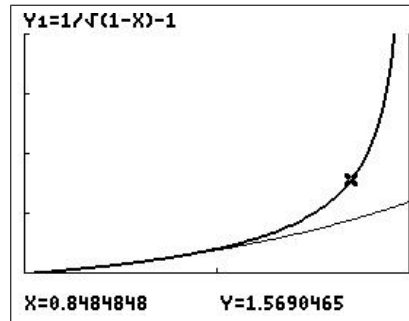
window $[0, 1, 0.5] \times [0, 4, 1]$

This function has a vertical asymptote at $x = 1$.

Also shown is the Taylor-series approximation up to the term in $x^3 = (v/c)^6$.

(c) Show that when $v \ll c$, the expression for K agrees with classical Newtonian Physics: $K(v) = \frac{1}{2}m_0v^2$.

If $v \ll c$, we can take just the first term in the series, giving

$$K(v) = m_0c^2 \cdot \frac{1}{2}\left(\frac{v}{c}\right)^2 = \frac{1}{2}m_0v^2.$$

(d) The second-order correction to the approximation $K \approx \frac{1}{2}m_0v^2$ is obtained by taking the next term in the series, but no higher terms. Use it to estimate the change in an object's kinetic energy for $v = 3440\,\mathrm{m/s}$ (Mach 10). Take $c = 3 \times 10^8\,\mathrm{m/s}$.

$$K \approx m_0c^2 \left(\frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4\right)$$

$$= \frac{1}{2}m_0v^2 \left(1 + \frac{3}{4}\left(\frac{v}{c}\right)^2\right).$$

When $v = 3440$, the second-order term, $\frac{3}{4}\left(\frac{v}{c}\right)^2 \approx 9.9 \times 10^{-11}$.

In percentage terms, the (relativistic) correction for Mach 10, due to the second-order term, is $9.9 \times 10^{-9}\,\%$, a very small number.

15.   *Physical Chemistry*

The potential energy V of two gas molecules separated by a distance r is given by

$$V(r) = -V_0 \left(2 \left(\frac{r_0}{r} \right)^6 - \left(\frac{r_0}{r} \right)^{12} \right),$$

where V_0 and r_0 are positive constants.

(a) Show that the global minimum of V, $V = -V_0$, occurs when $r = r_0$.

$$V(r) = -V_0 \left(2 \left(\frac{r_0}{r} \right)^6 - \left(\frac{r_0}{r} \right)^{12} \right),$$

with domain $r > 0$. The global minimum, if it exists, lies at a critical point.

$$\begin{aligned}
\frac{dV}{dr} &= -V_0 \left(12 \left(\frac{r_0}{r} \right)^5 \cdot \frac{-1}{r^2} - 12 \left(\frac{r_0}{r} \right)^{11} \cdot \frac{-1}{r^2} \right) \\[2mm]
&= \frac{12 V_0 r_0}{r^2} \cdot \left(\frac{r_0}{r} \right)^5 \left(1 - \left(\frac{r_0}{r} \right)^6 \right).
\end{aligned}$$

Therefore, dV/dr is defined for all $r > 0$ and is 0 when $r = r_0$.

Critical point: $V(r_0) = -V_0 \; < 0.$
'Endpoint':   As $r \to 0$, $V \to \infty$.
'Endpoint':   As $r \to \infty$, $V \to 0$.

Therefore, the global minimum $V = -V_0$ occurs when $r = r_0$.

(b) Sketch V/V_0 versus r/r_0 for $0.8 < r/r_0 < 1.2$.

$$\frac{V}{V_0} = \left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^6.$$

Set $Y_1 = 1/X^{12} - 2/X^6$, where $X = r/r_0$.
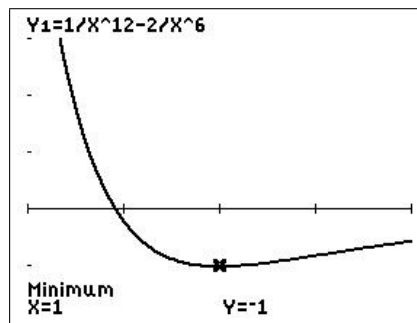
window $[0.8, 1.2, 0.1] \times [-1.2, 3, 1]$

PTO

(c) The force, F, between the molecules is given by $F = -dV/dr$. Derive the expression for the force and draw a graph of $F / \left(\frac{V_0}{r_0} \right)$ against r/r_0.

$$
\begin{aligned}
F(r) &= -\frac{dV}{dr} \\[2mm]
&= -\frac{12V_0 r_0}{r^2} \cdot \left(\frac{r_0}{r} \right)^5 \left(1 - \left(\frac{r_0}{r} \right)^6 \right), \quad \text{using the result from (a).}
\end{aligned}
$$

$$
\therefore \quad \frac{F}{(V_0/r_0)} = -12 \left(\frac{r_0}{r} \right)^7 \left(1 - \left(\frac{r_0}{r} \right)^6 \right).
$$

Set $Y_2 = -12/\mathrm{X}^{\wedge}7\left(1 - 1/\mathrm{X}^{\wedge}6\right)$: $\mathrm{X} = r/r_0$.
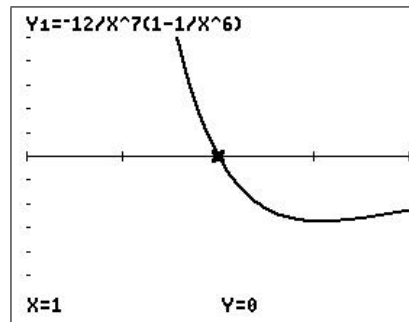
window $[0.8, 1.2, 0.1] \times [-5, 5, 1]$

(d) We note that a positive F represents a repulsive force; a negative F represents an attractive force. Is F repulsive or attractive when $r = r_0$? What does this mean physically?

At $r = r_0$, $F = 0$. The force is neither attractive nor repulsive: the molecules are in equilibrium.

(e) *Qualitatively*, what happens if the molecules start with $r = r_0$ and are pulled slightly further apart?

If the molecules are pulled apart, r becomes slightly greater than r_0 and so the force is negative (attractive): the molecules are pulled back together, towards equilibrium.

(f) *Qualitatively*, what happens if the molecules start with $r = r_0$ and are pushed slightly closer together?

If the molecules are pushed together, r becomes slightly less than r_0 and so the force is positive (repulsive): the molecules are forced apart, back towards equilibrium.

PTO

(g) The above expression for the force between the gas molecules is rather complicated and therefore not very useful. A Taylor series can help us develop a quantitative understanding of the nature of the force between the gas molecules when they are not too far from the equilibrium configuation. To do this, first write V as a Taylor series in $x = r/r_0$ centred at $x = 1$, retaining up to the quadratic term.

Write $V = -V_0(2x^{-6} - x^{-12})$, where $x = r/r_0$.

Find the Taylor series about $x = 1$ $(r = r_0)$ for $g(x) = 2x^{-6} - x^{-12}$.

$g(x) = 2x^{-6} - x^{-12} \qquad g(1) = 1$

$g'(x) = -12x^{-7} + 12x^{-13} \qquad g'(1) = 0$

$g''(x) = 84x^{-8} - 156x^{-14} \qquad g''(1) = -72$

Therefore,

$$g(x) \;=\; 1 \;-\; 72\,\frac{(x-1)^2}{2} \;+\; \cdots \;=\; 1 \;-\; 36(x-1)^2 \;+\; \cdots .$$

Therefore,

$$V \;=\; -V_0\big(1 - 36(x-1)^2 + \cdots\big) \;=\; -V_0\left(1 - 36\,\frac{(r-r_0)^2}{r_0^2} + \cdots\right).$$

(h) Differentiate this result to find a series representation for the force F.

$$F = -\frac{dV}{dr} \;=\; V_0\left(-\frac{72(r-r_0)}{r_0^2} + \cdots\right) \;=\; -\frac{72V_0(r-r_0)}{r_0^2} + \cdots .$$

(i) By discarding all except the first non-zero term in this series, describe how the force between the atoms depends on the displacement from the equilibrium when that displacement is small.

If $r > r_0$, F is negative, i.e. attractive. If $r < r_0$, F is positive, i.e. repulsive. This is what we found graphically in (e) and (f).

(j) Thus, for small displacements from the equilibrium, what sort of motion results?

The motion is always back towards the equilibrium position $r = r_0$, which is therefore a (locally) stable equilibrium.

13   Differential Equations

13.1   Introduction

The material here is based on lecture notes for a first-year university Calculus course at UNSW Canberra. It goes beyond the 'how do I do this?' of some of the other chapters in providing more context and modelling examples.

13.1.1   Why solve differential equations?

> Perhaps the most important of all the applications of Calculus is to differential equations. When physical scientists or engineers use Calculus, more often than not it is to analyze a differential equation that has arisen in the process of modelling some phenomenon that they are studying.
>
> Although it is often impossible to find an explicit formula for the solution of a differential equation, we will see that graphical and numerical approaches provide the needed information.
>
> <div align="right">James Stewart
Calculus: Concepts and Contexts</div>

13.1.2   Why graphics calculators?

Graphics calculators are a relatively cheap and easy-to-use way of implementing the *graphical and numerical approaches* referred to by James Stewart above. The programs for slope fields and Euler's Method are relatively simple and easy to relate to hand calculations. Both are also a good introduction to simple coding. Once you have drawn one or two slope fields or done several Euler calculations by hand, the advantages of a program become obvious. Any meaningful form of modelling without programs is impossible.

Euler's Method is a good introduction to numerical differential-equation solvers, their geometric basis and the important idea of accuracy. More-sophisticated methods are just elaborations of Euler's Method; an understanding of the basis of Euler's Method gives one intuition for the other methods and the general approach.

13.1.3   Modelling with differential equations

Scientists and engineers (and demographers, epidemiologists, economists, financial modellers, etc) use differential equations (DEs) in their modelling. That involves capturing the most important aspects of the system being modelled in a set of equations, solving the equations somehow and relating the findings back to the system. Here are some common models resulting in first-order DEs.

Population dynamics

In the absence of limiting factors, such as lack of food, biological populations tend to grow at a rate proportional to the total population, that is

$$\frac{dP(t)}{dt} \;=\; kP(t),$$

where $P(t)$ is the population at time t and k, the growth rate, is the constant of proportionality. Limiting factors, the presence of predators and other complexities can be added to the model to make it more realistic.

Other processes leading to a similar DE are radioactive decay, change of air pressure with height, compound interest, absorption of light and electrical circuits without capacitors.

Heating and cooling

Newton's Law of Cooling (and Heating)

The rate of change of the temperature T of a cooling body with time t is proportional to the difference between the temperature of the body and that of its surroundings, T_s. Mathematically, this is written as

$$\frac{dT(t)}{dt} = -k\bigl(T(t) - T_s\bigr),$$

where k is the proportionality constant.

Other processes leading to a similar DE are mixing of substances in tanks, concentration of drugs in the body and simple learning models.

Projectile motion

Newton's Second Law of Motion:

$$\text{mass} \times \text{acceleration} = \text{sum of forces acting on the body.}$$

Acceleration is the time rate of change of velocity $\dfrac{dv}{dt}$. The DE which describes a simple model of a body falling under gravity with air resistance (drag) is

$$m\frac{dv}{dt} = mg - kv^n,$$

where m is the mass and v the velocity of the body, k is the coefficient of air resistance and n is a constant that depends on the speed and the size of the body. mg is the gravitational force and kv^n the drag or air-resistance force.

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13.2   First-order DEs: A graphical method — slope fields

The slope field of a first-order DE is a grid of short line segments on the Cartesian (xy) plane, with the slope of the line segment at a particular point being the slope of the (unique) solution to the DE that passes through that point; this slope (derivative) is given by the DE. Slope fields are useful in showing the global behaviour of solutions to a DE and require no knowledge of methods of solving DEs.

Before the ready availability of graphics, slope fields had to be drawn by hand, an easy but time-consuming process. The use of isoclines, lines of equal slope, helped a little but no-one was going to draw many slope fields. With the use of a relatively simple program, graphics calculators can draw slope fields quickly and accurately. By changing the calculator window, you can examine particular regions of the solutions in greater detail or 'zoom out' for the big picture.

The calculator program SLPFIELD/SLPFLDCE draws a slope field for a DE in the form

$$\frac{d\text{Y}}{d\text{X}} \;=\; f(\text{X}, \text{Y}).$$

Use: Press $\boxed{\text{y=}}$ and enter the function $f(\text{X}, \text{Y})$, the right-hand side of the DE, in Y₁. This function will not be graphed (and cannot if it contains Y) but is evaluated by the program at each grid point to find the derivative/slope there.

Choose a suitable $\boxed{\text{window}}$, press $\boxed{\text{prgm}}$, select and run the program.

Example: Plot the slope field for the differential equation

$$\frac{dy}{dx} = x - y$$

over the region $-4 \leqslant x \leqslant 4$, $-4 \leqslant y \leqslant 4$. Sketch the curve of the solution of the DE that passes through the point $(0,0)$, that through $(1,0)$ and that through $(2,0)$.

Set Y₁ $=$ X$-$Y and a $\boxed{\text{window}}$ of $[-4, 4, 1] \times [-4, 4, 1]$.

```
                                    WINDOW
                                     Xmin=-4
                                     Xmax=4
                                     Xscl=1
                                     Ymin=-4
                                     Ymax=4
                     ■\Y₁=X−Y        Yscl=1
```
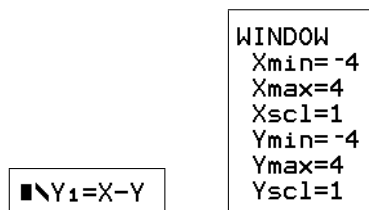
Run the program to give the slope field below. This shows three different types of solution curves: concave-up solutions in most of three of the four quadrants; concave-down solutions in most of the fourth quadrant; and what looks like a straight-line solution running from bottom left to top right and separating the other two types.
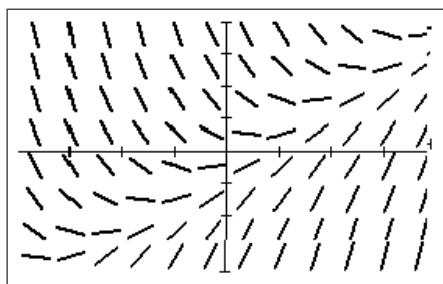
Following the slope lines, draw the solutions that pass through $(0,0)$, through $(1,0)$ and through $(2,0)$.
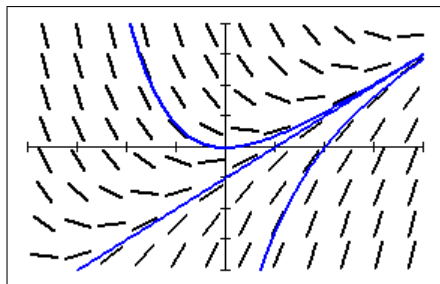
One way to do this is generate the slope field on your calculator, download the screenshot to your computer using TI-Connect/CE and print it out so you can draw in the solutions.

You can plot solutions on the slope field (as I have done here) using one of the DE solvers in the next section; just remove the *ClrDraw* command near the start of the relevant program.

Exercises

Solutions to all exercises are in Section 13.8.

1. Sketch a slope field for the differential equation $\dfrac{dy}{dx} = y - 2x$.

   Then use it to draw the solution curve that passes through the point $(1,0)$.

   Use a window of $[-3,3,1] \times [-2,2,1]$.

2. Sketch a slope field for the differential equation $\dfrac{dy}{dx} = x - xy$.

   Then use it to draw the solution curve that passes through the point $(1,0)$.

3. Sketch a slope field for the differential equation $\dfrac{dy}{dx} = 2 - y$.

   Then use it to draw three representative solution curves, i.e. curves showing the different types of behaviour of the solutions.

   Use a window $[0,3,1] \times [0,4.5,1]$.

13.3   First-order DEs: Numerical methods

13.3.1   Euler's method

Rather than trying to follow the slope lines in a slope field in drawing an approximate solution curve, it would seem to make sense to draw in the necessary slope lines as we go. This simple idea is the basis of Euler's method.

The method

Example: *Using Euler's method — a simple case*

Use Euler's method to plot an approximation to the solution $y(x)$ of the initial-value problem

$$\frac{dy}{dx} = 2x \qquad y(-2) = 4.$$

Take $-2 < x < 2$.

This is a simple case that can be done easily algebraically; we use it here to understand Euler's Method. The exact (algebraic) answer can be used as a check.

We are going to construct an approximate solution to this initial-value problem. The solution curve will consist of a number of slope lines joined end to end.

In drawing the slope lines, we need to specify how long they are to be. The shorter they are, the smoother the curve and the closer the approximate curve will be to the exact one, but the more calculations we shall have to do.

It turns out to be easier to specify not the length of the slope line but the change in x from one end of the slope line to the other.

In drawing a slope line, we start at a given point (x_0, y_0), and draw in the slope line of slope m, calculated at the point (x_0, y_0) from the DE. By definition, $m = \Delta y / \Delta x$.

Along the slope line, if x changes by a given amount called the step length $h = \Delta x$, the change in y is given by $\Delta y = m\Delta x = mh$.

The next point on our approximate curve is then $(x_1, y_1) = (x_0 + h,\ y_0 + mh)$.

In effect, we are approximating the solution curve on the interval $x_0 \leqslant x \leqslant x_0 + h$ by a linear function, the tangent to the curve at (x_0, y_0).

To continue, we repeat the whole process starting now at the new point (x_1, y_1).

For our example, we start with a step length $h = 0.5$. The starting point for our example is the initial condition $(x_0, y_0) = (-2, 4)$.

The EULER1/EULER1CE program

As with slope fields, this calculation is tedious by hand, but is easily programmed, as in the EULER1/EULER1CE program. To use this program, first write the DE in terms of the calculator variables X (independent variable) and Y (dependent variable), as we did for SLPFIELD/SLPFLDCE:

$$\frac{d\text{Y}}{d\text{X}} = f(\text{X}, \text{Y}).$$

- Press $\boxed{\text{y=}}$ and set Y1 = the right-hand side of the DE: in this example Y1 = 2X.

```
              WINDOW
               Xmin=-2
               Xmax=2
               Xscl=1
               Ymin=-2
               Ymax=4
  ■\Y₁■2X      Yscl=1
```
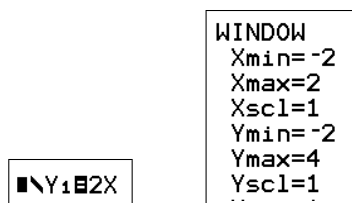
- Set the $\boxed{\text{window}}$ parameters to suitable values (so we see the whole graph):
  $[-2, 2, 1] \times [-2, 4, 1]$ is about right here.

  The initial X value gives Xmin.

  As with plotting graphs, you have to experiment a bit with the Y window, but the initial Y value helps.

  The two steps above are the same as for SLPFIELD/SLPFLDCE.

- Press $\boxed{\text{prgm}}$, select and run the EULER1/EULER1CE program.

  Enter the initial point, $(X0, Y0) = (-2, 4)$ here; the step H, here 0.5; and press $\boxed{\text{enter}}$. The calculator should then draw in the whole curve from Xmin to Xmax.

```
        EULER1            [ENTER]        ▮NEXT...
                                         1▮NEW H
  INITIAL POINT                          2:NEW X0,Y0,H
  X0: -2                                 3:FINAL POINT
  Y0:4                                   4:PLOT AT CURSOR
                                         5:CLEAR GRAPHS
  STEP H:0.5         X=0        Y=1      6:QUIT
```
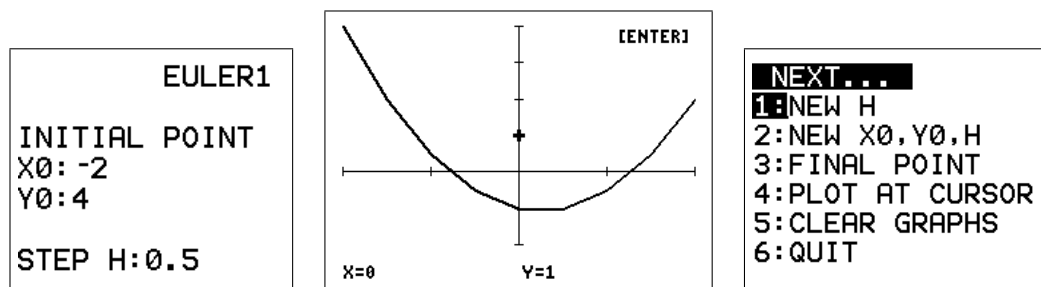
- Press $\boxed{\text{enter}}$ to go to the menu where you can choose to enter a new H value, a new initial point or one of several other options. Selecting PLOT AT CURSOR allows you to move the cursor to any point that you want a solution to pass through; pressing $\boxed{\text{enter}}$ then causes the solution to be plotted.

- Select $\boxed{6}$ (QUIT) to stop the program.

Exercise: *Changing the step length*

Try some smaller H values, say 0.1, 0.01. The value 0.5 in the example is relatively large. What do you observe? *Solutions in Section 13.8.*

Notes

- Intuitively, as h gets smaller, the Euler approximation to the solution $y(x)$ gets better.

  If you magnify a differentiable function sufficiently at any point, it looks like a straight line, the tangent line at that point. Therefore, approximating the function y **on each subinterval** $[x, x+h]$ by a straight line should become more and more accurate as h (the width of the subinterval) gets smaller.

- In the limit $h \to 0$, the *approximate* solution generated by Euler's method tends to the *exact* solution y.

- We can get closer and closer to the exact solution by taking h smaller and smaller, but as you have observed, it takes the calculator longer and longer to do the calculations.

- Ultimately we are limited in our accuracy by round-off error. If h is too small, the number of digits carried by the calculator (14) will not be sufficient to generate an accurate answer. Therefore *all* numerical solutions are approximations to the exact solution.

Exercise: *Changing the initial point*

Now experiment with different initial Y values. Take $H = 0.1$. What do you observe?
Solutions in Section 13.8.

It's a good idea to clear the graphs (menu item) or stop the program and restart it at this point, otherwise the screen gets cluttered.

Notes

- In the example above, we know that the general solution of the DE is $y = x^2 + C$, where C is a constant. Different initial points will give different C values, hence generating a family of parallel curves displaced from one another along the y axis, as you should have found in the exercise above.

- The value of C is determined by specifying a point on the curve, here the initial point $(-2, 4)$:
  $$y(-2) = 4 \;\Rightarrow\; 4 = (-2)^2 + C = 4 + C \;\Rightarrow\; C = 0.$$
  The particular solution is then $y = x^2$. Compare with the Euler approximations.

  Calculator tip: If you know the exact solution $y(x)$, you can superimpose it on the graphs of the approximations generated by Euler's method. Quit the program, put the exact solution in Y₂ and press $\boxed{\text{graph}}$.

Example: *Numerical calculations using Euler's method*

Find $y(2)$ if
$$\frac{dy}{dx} = \frac{2y}{x} \quad \text{and} \quad y(1) = 3.$$

Set Y₁ $= 2$Y/X; set the $\boxed{\text{window}}$ to $[1, 2, 0.5] \times [-2, 12, 2]$.

Note: Xmin should be set to the initial x value x_0, so Xmin $= 1$ here.

If we set up the $\boxed{\text{window}}$ so that Xmax is the X value of the point we want (2 here), pressing $\boxed{\text{enter}}$ $\boxed{3}$ *after the graph is plotted will display the required Y value.*

Note: There has to be an integral number of step lengths H between Xmin/x_0 and Xmax for the Y value at Xmax to be displayed.

Run EULER1/EULER1CE with an initial X of 1 and an initial Y of 3 (the initial condition), and step length[26] H $= 0.1$ to see what the solution looks like.

Press $\boxed{\text{enter}}$ $\boxed{3}$ to display the last values of X and Y calculated (menu option 3).

---

[26]H $= 0.1$ gives 10 steps from Xmin to Xmax, a reasonable number of steps to start with.

Exercise: Calculate $y(2)$ in this manner, reducing H in powers of 10. Round values to 4 decimal places and enter them in the table below.

H	$y(2)$
0.1	11.4545
0.01	
0.001	

What's the best estimate for $y(2)$ from EULER1/EULER1CE?

Round off the last two y values (whose H values differ by a factor of 10) until they are the same. That tells you the exact y value (in most cases), **accurate to that number of digits**.

Here the two y values are 11.9406 and 11.9940. We have to round these back to 2 significant digits before they are the same.

Hence, from these calculations, $y(2) = 12$, accurate to 2 significant digits.

Euler's method is equivalent to the Left-Endpoint Rule for numerical integration.

13.3.2   Modified Euler's method

The program MODEULR1/MEULR1CE uses the modified Euler or Heun method. The slope of the slope line it draws at each step is the average of the slopes at each end of the line that would be drawn using Euler's method. You run MODEULR1/MEULR1CE the same way you run EULER1/EULER1CE.

MODEULR1/MEULR1CE is more accurate than EULER1/EULER1CE for a given H value, and therefore produces an answer accurate to a given number of significant digits faster than EULER1/EULER1CE does. The modified Euler's method is equivalent to the Trapezoidal Rule for numerical integration.

Exercise: Use MODEULR1/MEULR1CE to complete the table below, finding $y(2)$ if

$$\frac{dy}{dx} = \frac{2y}{x} \quad \text{and} \quad y(1) = 3.$$

What's your best estimate for $y(2)$ from MODEULR1/MEULR1CE? How many significant digits?

H	$y(2)$
0.1	11.9600
0.01	
0.001	

Stop the program by selecting Quit, set $Y_2 = X^2$ and press $\boxed{\text{graph}}$.

How accurate are our answers?

If two consecutive y values in the table (with H values differing by a factor of 10) are the same when rounded to n significant digits, we can be *reasonably* sure that this is the exact answer, <u>accurate</u> to n significant digits.

Therefore, comparing the last two answers in each table:

EULER1/EULER1CE: $y(2) = 12$, accurate to 2 significant digits.

MODEULR1/MEULR1CE: $y(2) = 12.000$, accurate to 5 significant digits.

The numbers generated by Euler's method appear to converge, albeit slowly, as H becomes smaller. When $H = 0.001$, the calculation takes a long time (it has to do 1000 steps), and the result is still only accurate to 2 significant digits. If we wanted a value of y for x larger than 2, we would have to take still smaller step sizes to achieve the same accuracy.

Therefore, although in principle we can achieve any given accuracy using Euler's method by taking a small enough step size (providing round-off error does not become significant), the time taken to do the calculations can become large.

We then look to methods that converge faster, such as the modified Euler's method (MODEULR1/MEULR1CE) or Runge-Kutta methods (Section 13.3.3). For graphing and numerical purposes over not-too-large ranges of x, however, EULER1/EULER1CE is OK.

Computer packages such as Matlab use fancier methods, such as Runge-Kutta, but the basic idea is the same.[27]   If you understand Euler's method, you will have a good grasp of the others.

PTO

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[27]essentially, they use a weighted average of values of the derivative/slope at various x values as the slope of the line to the next point

Exercises

Solutions in Section 13.8.

1. Consider the differential equation $\dfrac{dy}{dx} = x - y$, with initial condition $y(0) = 1$.

   (a) Use EULER1/EULER1CE to find $y(0.4)$ *accurate* to 2 significant digits.[28] Set up a
   table of values of $y(0.4)$ versus H, like in the exercises above.

   (b) How do you *know* each answer in (a) is *accurate* to 2 significant digits?

   (c) Use MODEULR1/MEULR1CE to find $y(0.4)$ *accurate* to 4 significant digits.

   (d) The exact solution to the DE for which $y(0)=1$ is

   $$y(x) = x - 1 + 2e^{-x}.$$

   How does the error in using EULER1/EULER1CE to calculate $y(0.4)$ vary with H
   in this case? You may need to calculate $y(0.4)$ for a few more H values and keep
   more digits. Add a column to your table for the error in using Euler's method.

   What return in improved accuracy does Euler's method give for the increased work
   caused by reducing the step size H by a factor of 10?

   What about the modified Euler's method? (Keep 6 digits in your values here.)

2. **(a)** Find $P(3)$ and $P(12)$, accurate to 3 significant digits, if $P(0) = 1$ and

   $$\frac{dP}{dt} = 0.3e^{-0.1t^2}P.$$

   Hint: Use MODEULR1/MEULR1CE; write the DE in calculator variables; set up
   a table of $P(3)$ and $P(12)$ vs H to determine when you have reached the required
   accuracy.

   (b) What is the approximate value of $P(t)$ as $t \to \infty$?

3. **(a)** Use Euler's method to find $y(2)$, accurate to 3 significant digits, if $\dfrac{dy}{dx} = \dfrac{1}{x}$ and
   $y(1)=0$.

   (b) What step length H do you need in MODEULR1/MEULR1CE to obtain an answer
   accurate to 5 significant digits?

4. **(a)** Find $y(1.5)$ accurate to 3 significant digits for the initial-value problem

   $$y' + \tan(x)\,y = \cos^2(x) \qquad y(0) = -1$$

   using MODEULR1/MEULR1CE. *Is your calculator in RADIAN mode?* It should
   be, always.

   (b) The algebraic (exact) solution to the initial-value problem here (on the interval
   $0 < x < \pi/2$) is $y(x) = \sin(x)\cos(x) - \cos(x)$. Use this to check that your answer
   to (a) is indeed accurate to 3 significant digits.

---

[28]Rounded to 2 significant digits is not the same as accurate to 2 significant digits.

13.3.3 Runge-Kutta methods

Runge-Kutta methods use a weighted average of the derivative/slope at various x values in each interval $[x_i, x_i+h]$ as the slope of the line to the next point; a fourth-order Runge-Kutta method uses four such values. For each reduction of H by a factor of 10, we achieve a reduction in the error of about a factor of 10^4 (see the Exercises below).

The Runga-Kutta suite of programs uses a fourth-order Runge-Kutta method to solve first-order DEs. These programs work in the same way as MODEULR1/MEULR1CE but come in two forms: one that graphs the solution (RK1GRAPH/RK1GPHCE) and a faster one that just calculates the final value (RK1CALC/RK1CALCE).

Example: Find $y(2)$ if

$$\frac{dy}{dx} = \frac{2y}{x} \quad \text{and} \quad y(1) = 3.$$

What's your best estimate for $y(2)$ from RK1GRAPH/RK1GPHCE? How many significant digits?

As with the other programs, set $Y_1 = 2Y/X$, set the $\boxed{\text{window}}$ to $[1, 2, 0.5] \times [-2, 12, 2]$ and $X0 = 1$, $Y0 = 3$.

H	$y(2)$
0.1	11.99993823
0.01	11.99999999
0.001	12.00000000

Then, $y(2) = 12.0000000$, accurate to 9 significant digits. Compare with the values from Euler's method and the modified Euler's method for the same H in Sections 13.3.1 and 13.3.2. Clearly, this method is superior, and the one to use for day-to-day calculations.

This fourth-order Runge-Kutta method is equivalent to Simpson's Rule for numerical integration.

Exercises

Repeat the exercises on page 140 using the fourth-order Runge-Kutta method instead of the modified Euler's method.

Compendium program

The DE1CALC/DE1CALCE program calculates final Y values, given a DE, an initial point, a final X value and an H value. You can choose which of the three methods, Euler's method, the modified Euler's method or the fourth-order Runga-Kutta method, to use.

Group file: First-order DEs

The group file DE1ORD.8xg/DE1ORDCE.8xg contains all the first-order DE solvers (see Section 16.2.3).

13.4 Systems of first-order DEs

13.4.1 Two coupled DEs

Both Euler's and modified Euler's methods[29] are easily adapted to systems of differential equations. The modified Euler's method in this case is related to Euler's method exactly as it was for a single first-order DE. Again, the modified Euler's method is the better method to use because it gives a better accuracy for a given step size. Runge-Kutta is even better.

We illustrate the modified Euler's method using the MODEULR2/MEULR2CE program with the following simple example. EULER2/EULER2CE does the same thing using Euler's method as does RK2GRAPH/RK2GPHCE using the fourth-order Runga-Kutta method.

Example: Solve the following system of coupled differential equations numerically, subject to the given initial conditions.

$$\frac{dy_1}{dx} = -y_1 + y_2 \tag{1}$$

$$\frac{dy_2}{dx} = 2y_1 \tag{2}$$

$$y_1(0) = 0 \qquad y_2(0) = 1.$$

Plot the solution curves $y_1(x)$ and $y_2(x)$ for $0 < x < 2$.

Find $y_1(2)$ and $y_2(2)$ accurate to 3 significant digits.

Using the MODEULR2/MEULR2CE program

The program MODEULR2/MEULR2CE does for a system of two first-order differential equations what MODEULR1/MEULR1CE did for a single first-order differential equation. It too works for a system in which the right-hand sides of the DEs are **non-linear**.

The MODEULR2/MEULR2CE program assumes a system of first-order DEs of the form

$$\frac{d\text{U}}{d\text{X}} = \text{Y}_1\,(\text{X, U, V}) \qquad \frac{d\text{V}}{d\text{X}} = \text{Y}_2\,(\text{X, U, V}), \tag{3}$$

that is independent variable X, dependent variables U and V.

We therefore have to translate the differential equations in our problem to **calculator variables** X, U, V.

The right-hand side of the first DE, Eq. (1), is stored in Y$_1$ on the calculator in terms of U and V; the right-hand side of the second DE, Eq. (2), in Y$_2$.

U and V are just generated by $\boxed{\text{alpha}}\ \boxed{\text{U}}$ and $\boxed{\text{alpha}}\ \boxed{\text{V}}$.

PTO

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[29]and the fourth-order Runge-Kutta method — see Section 13.3.3

Step 1: Convert from problem variables to calculator variables

Comparing Eqs. (1) and (2) with the calculator equations, Eq. (3), we see that, in this case,

$$x \to \mathrm{X} \qquad y_1 \to \mathrm{U} \qquad y_2 \to \mathrm{V}.$$

Therefore (using the correct minus on your calculator),

$$\frac{dy_1}{dx} \;=\; -y_1 \,+\, y_2 \quad \to \quad \frac{d\mathrm{U}}{d\mathrm{X}} \;=\; -\mathrm{U} + \mathrm{V}. \qquad \text{Set } \mathrm{Y_1} = -\mathrm{U{+}V}.$$

$$\frac{dy_2}{dx} \;=\; yx_1 \quad \to \quad \frac{d\mathrm{V}}{d\mathrm{X}} \;=\; 2\mathrm{U}. \qquad \text{Set } \mathrm{Y_2} = 2\mathrm{U}.$$

```
                                      WINDOW
                                       Xmin=0
                                       Xmax=2
                                       Xscl=1
                                       Ymin=-1
             ■\Y₁⬛ -U+V                 Ymax=5
             ■\Y₂⬛2U                    Yscl=1
```

Step 2: Set the window for the solution

We need to tell the program the range of X (*x*) values for which we want the solution. We do this by setting the window parameters.

For this problem, set the window to $[0, 2, 1] \times [-1, 5, 1]$. Thus, we are solving the DEs for $x = 0$ to $x = 2$, as stated in the problem. The Y (U and V) range is found by experimenting and taking into account the initial conditions.

Step 3: Running the program

Now run the program MODEULR2/MEULR2CE .

```
                            TIME PLOT

                     Xmin=0

   MODEULER2        U(Xmin): 0               GRAPH
 1:TIME PLOT        V(Xmin): 1             1:U(X)
 2:PHASE PLOT       STEP H: 0.1             2:V(X)
 3:QUIT
```

- Choose TIME PLOT.[30]

- Set $\mathrm{U(Xmin)} = 0 \quad \mathrm{U}(0) = y_1(0) = 0$.

- Set $\mathrm{V(Xmin)} = 1 \quad \mathrm{V}(0) = y_2(0) = 1$.

- Set STEP $\mathrm{H} = 0.1$ initially to give a reasonable number of steps between $x = 0$ and $x = 2$.

- Choose the graph of U(X), that is $y_1(t)$ in this case.

---

[30]A time plot is U and/or V vs X; a phase plot is V vs U.

The program then plots an (approximate) modified Euler solution for the system.
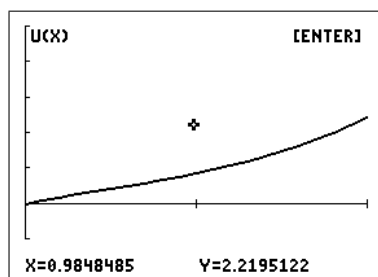
Step 4: The results

Pressing ⌷enter⌷ after the graph has been plotted to see the final values calculated.[31]

Pressing ⌷enter⌷ once more gives a range of options: new H; new U(Xmin) and V(Xmin); plotting the other solution V/U; clearing the graphs before continuing; changing the window; and quitting.

How accurate is the solution?

We proceed exactly as we did with MODEULR1/MEULR1CE.

We write down the (approximate) values, given by the program, for $y_1(2)$ (U) and $y_2(2)$ (V) with a step size of H = 0.1 in the table below. Then repeat with H = 0.01 and, in this case to achieve the desired accuracy, H = 0.001 (have a think while this one is running).

The figure below shows the graphs of both U and V for the three H values. The graphs for all three H values are close, the graphs for H = 0.01 and H = 0.001 almost indistinguishable.

H	$y_1(2)$	$y_2(2)$
0.1	2.45	4.92
0.01	2.46	4.93
0.001	2.46	4.93
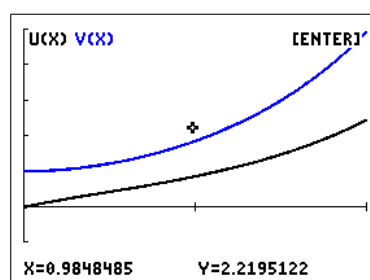
We conclude from the values in the table that $y_1(2) = 2.46$ and $y_2(2) = 4.93$, both accurate to 3 significant digits.

ME2CALC/ME2CALCE just calculates the final U and V values without the graphs.

We use this method again in Section 13.5, where we rewrite a second-order differential equation as two coupled first-order differential equations.

---

[31]You only have to plot one of U or V to see all the calculated values.

RK2CALC/RK2CALCE uses the fourth-order Runga-Kutta method to calculate final values for two coupled DEs. It works the same way as MODEULR2/MEULR2CE: see page 142.

Group file: DE2ORD.8xg/DE2ORDCE.8xg contains all the solvers for two coupled DEs (see Section 16.2.3).

Exercises   *Solutions are in Section 13.8.*

1. Plot the solutions to the following system on $0 \leqslant x \leqslant 2$, with initial conditions $y_1(0) = 8$ and $y_2(0) = 2$:

$$\frac{dy_1}{dx} = -2y_1 + 3y_2 + 12e^x$$

$$\frac{dy_2}{dx} = y_1 - 4y_2.$$

   Find $y_1(2)$ and $y_2(2)$ accurate to 4 significant digits.

2. Let $u_P(t)$ be the amount (mass) of a drug in the blood plasma at time t days, and let $u_T(t)$ be the amount in the tissue (organs in the body) for certain positive rate constants k_a, k_b and k_e. The following equations model the process in which the drug enters the plasma at rate $g(t)$, moves between the plasma and the tissue, and is excreted from the plasma:

$$\frac{du_P}{dt} = -k_b u_P - k_e u_P + k_a u_T + g(t)$$

$$\frac{du_T}{dt} = k_b u_P - k_a u_T.$$

   Consider the case in which $k_a = 4$, $k_b = 2$, $k_e = 3$ and $g(t) = 0$, giving equations

$$\frac{du_P}{dt} = -5u_P + 4u_T$$

$$\frac{du_T}{dt} = 2u_P - 4u_T.$$

   Suppose the initial conditions are $u_P(0) = 1$ and $u_T(0) = 0$: a dose of 1 unit of a drug is injected into the plasma or bloodstream at time $t = 0$, with none in the tissue. Plot $u_P(t)$ and $u_T(t)$ on the interval $0 \leqslant t \leqslant 2$ and comment: do these graphs make sense for the amount of drug in the plasma and tissue?

   From Section 6.1 of *Differential Equations: A Toolbox for Modeling the World* by Kurt Bryan, SIMIODE, Cornwall, New York.

13.4.2   Three coupled DEs

The MODEULR3/MEULR3CE program assumes a system of first-order DEs of the form

$$\frac{d\mathrm{U}}{d\mathrm{X}} = \mathrm{Y}_1\left(\mathrm{X,U,V,W}\right) \qquad \frac{d\mathrm{V}}{d\mathrm{X}} = \mathrm{Y}_2\left(\mathrm{X,U,V,W}\right) \qquad \frac{d\mathrm{W}}{d\mathrm{X}} = \mathrm{Y}_3\left(\mathrm{X,U,V,W}\right),$$

that is independent variable X, dependent variables U, V and W.

The program works in exactly the same way as MODEULR2/MEULR2CE but with the extra dependent variable W. The ME3CALC/ME3CALCE program just calculates the final U, V and W values.

13.4.3   Up to twenty coupled DEs

Mathematicians like to generalise: having written programs to solve numerically systems of 2 and 3 coupled DEs, could I write a program for a system of n coupled DEs with, initially, $n \leqslant 10$, restricted by the number of functions in $\boxed{\mathrm{y=}}$? Then, what about $n \leqslant 20$?

These programs were also written to show that a relatively simple device such as a graphics calculator could solve (numerically) large systems of DEs. The suite of programs is DE20PLCE/DE20CLCE/DE20INPT/DE20SUB for the 84CE,[32] which solves (plots solutions/ calculates values) a system of up to 20 coupled DEs by a choice of the three methods.

In moving to n coupled first-order DEs, a change in the dependent variables is necessary. They can no longer easily be represented by separate letters, as was the case above. Here, the values of the dependent variables at each step in the process are stored in a list (vector) ʟY; the length of this list varies as the number of DEs/dependent variables varies, which is why *up to* 20 DEs can be specified.

The system of equations can then be written in vector form as

$$\frac{d\,\mathrm{\text{ʟY}}}{dx} = \mathbf{R}(\mathrm{\text{ʟY}}),$$

where $\mathbf{R}$ is a vector function whose components are the right-hand sides of the DEs, each of which is a function of X and the components of ʟY in general.

Therefore, in putting the right-hand sides of the DEs into $\mathrm{Y}_1 - \mathrm{Y}_0$ (for the first 10 DEs), the independent variable is still X but the dependent variables are ʟY(1), ʟY(2), ʟY(3), ..., ʟY(10), as many as are required. The SET UP FIRST option in the first menu of the main programs establishes the list ʟY, so that its name can be recalled from the $\boxed{\mathrm{list}}$ NAMES menu when putting in the DEs. More than 10 DEs requires a different approach and a separate program, DE20INPT, to store the extra DEs.

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[32]DE20PLOT/DE20CALC/DE20INPT/DE20SUB for the 84Plus

Example: Solve the following system of four coupled first-order DEs for $0 < x < 2$ with initial conditions as specified. Find values for the dependent variables at $x = 2$ accurate to 4 decimal places.

$$\frac{dy_1}{dx} = -y_1 + 0.1y_2$$

$$\frac{dy_2}{dx} = 0.1y_1 - y_2 - 0.1y_3$$

$$\frac{dy_3}{dx} = 0.1y_1 + y_2 - y_3 - 0.1y_4$$

$$\frac{dy_4}{dx} = y_3 - y_4$$

$$y_1(0) = 1 \qquad y_2(0) = 0.8 \qquad y_3(0) = 0.6 \qquad y_4(0) = 0.4$$

The first step is to rewrite the DEs in terms of calculator variables (actually, we only need the right-hand sides for the program).

$d\,\text{\tiny L}Y(1)/dX = -\text{\tiny L}Y(1) + 0.1\text{\tiny L}Y(2)$

$d\,\text{\tiny L}Y(2)/dX = 0.1\text{\tiny L}Y(1) - \text{\tiny L}Y(2) - 0.1\text{\tiny L}Y(3)$

$d\,\text{\tiny L}Y(3)/dX = 0.1\text{\tiny L}Y(1) + \text{\tiny L}Y(2) - \text{\tiny L}Y(3) - 0.1\text{\tiny L}Y(4)$

$d\,\text{\tiny L}Y(4)/dX = \text{\tiny L}Y(3) - \text{\tiny L}Y(4)$

$\text{\tiny L}Y(1)(0) = 1 \quad \text{\tiny L}Y(2)(0) = 0.8 \quad \text{\tiny L}Y(3)(0) = 0.6 \quad \text{\tiny L}Y(4)(0) = 0.4$

Run DE20CLCE, press ⌷enter⌷ and ⌷1⌷ for SET UP FIRST, then ⌷enter⌷ again and ⌷2⌷ for QUIT TO ENTER DES.

```
        DE SOLVER
   COUPLED 1ST-ORDER DES

VARS X,ʟY(I),I=1,2,....,20.
RHS OF DES: Y₁,Y₂,....,Y₀,
 Str1,Str2,....,Str0.
 ʟY(1)'=Y₁,ʟY(2)'=Y₂,....,
 ʟY(11)'=Str1,ʟY(12)'=Str2
NEEDS DE20SUB.
                    [ENTER]
```

```
▌DE20CALC▐
█:SET UP FIRST
2:RUN
3:QUIT
```

```
     SETUP COMPLETE

LIST NAME ʟY NOW AVAILABLE
TO USE IN INPUTTING RHS OF
     DES INTO Y₁,Y₂,....



                    [ENTER]
```

```
▌NEXT...▐
1:RUN PROGRAM
█:QUIT TO ENTER DES
```

Press $\boxed{\text{y=}}$ and enter the right-hand sides of the DEs as shown below left.

```
■\Y₁=⁻ʟY(1)+0.1ʟY(2)
■\Y₂=0.1ʟY(1)-ʟY(2)-0.1ʟY(
3)
■\Y₃=0.1ʟY(1)+ʟY(2)-ʟY(3)-
0.1ʟY(4)
■\Y₄=ʟY(3)-ʟY(4)
```

```
        INPUT PARAMETERS

NUMBER OF DES (1-10)
N=?4
Xmin=?0
Xmax=?2
INITIAL Y VECTOR ʟY(Xmin)
START WITH {
?{1,0.8,0.6,0.4}
STEP H: 0.1
```

Re-run the program and choose RUN. Enter the values at the prompts, as shown above right. Note that the initial conditions are entered as a list (vector), requiring a starting curly bracket; the closing curly bracket is optional.

Choose the method: RUNGE-KUTTA 4 here (below left). The order of each method is given.

```
 METHOD 
1:RUNGE-KUTTA 4
2:MODEULER 2
3:EULER 1
```

```
       RUNGE-KUTTA 4
H=0.1
INITIAL VALUES
X=0
ʟY={1,0.8,0.6,0.4}
FINAL VALUES
X=2
ʟY(X)= (4DP)
{0.1567 0.0996 0.2825 0.4…
   [L/R ARROWS] [ENTER]
```

The final values for the dependent variables, rounded to 4 decimal places, are given as a list (vector); use the arrow keys to scroll along the list (above right).

Pressing $\boxed{\text{enter}}$ gives a menu of options:

```
 NEXT... 
1:NEW H
2:NEW ʟY(Xmin)
3:NEW Xmin,Xmax
4:NEW METHOD SAME X0,ʟY0
5:NEW METHOD NEW X0,ʟY0
6:QUIT
```

To determine the final values to an accuracy of 4 decimal places, we need to rerun the program with at least H = 0.01. The results are shown in the table below.

H	$y_1(2)$	$y_2(2)$	$y_3(2)$	$y_4(2)$
0.1	0.1567	0.0996	0.2825	0.4328
0.01	0.1567	0.0996	0.2825	0.4328

Therefore, the values given with H = 0.1 are accurate to at least 4 decimal places.

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Running DE20PLCE to plot the solutions for the same problem requires the $\boxed{\text{window}}$ to be set first: here $[0, 2, 1] \times [0, 1, 0.1]$. The program screens are essentially the same as above until the plotting stage, when you have to choose which solutions to plot:
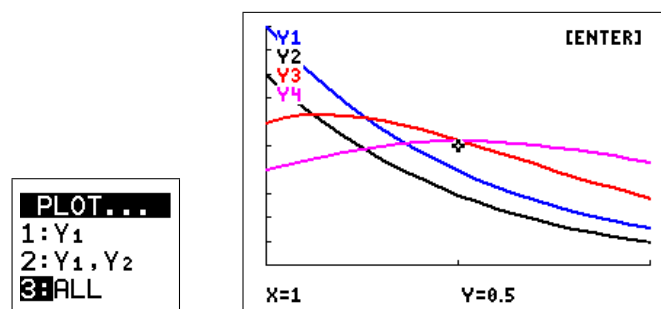
Choosing ALL gives the plot above right. The different curves come out in different colours (with labels in the same colour) and/or line styles.

The 84Plus version, DE20PLOT, only has the first two options in the PLOT menu because the 84Plus will only draw two line types.

The DE20PLCE/DE20CLCE/DE20INPT/DE20SUB suite of programs and the 84Plus equivalent have the extra input program DE20INPT to allow the right-hand sides of DEs 11, 12, ..., 20 to be input. As there are no more $\boxed{\text{y=}}$ functions available for this, they are stored in strings Str1 – Str0.
In the calculations, these are evaluated as functions in DE20SUB (using the command *expr*) to give the corresponding derivative.

The group file DE20.8xg/DE20CE.8xg contains the four programs here (see Section 16.2.3).

I can't imagine these programs ever being used but it was fun to write them. If you find an application, please let me know (pdmcintyre@icloud.com).

13.5   Second-order DEs

13.5.1   Theory

We convert a second-order DE to two coupled first-order DEs by defining a new dependent variable. If the second-order DE is

$$\frac{d^2u}{dx^2} + a\frac{du}{dx} + bu = f(x),$$

let

$$v = \frac{du}{dx}. \tag{4}$$

Then, $\frac{dv}{dx} = \frac{d^2u}{dx^2}$, and the second-order DE becomes

$$\frac{dv}{dx} + av + bu = f(x). \tag{5}$$

Equations (4) and (5) are a first-order system of two coupled DEs in dependent variables u and v that we can solve numerically with, for example, MODEULR2/MEULR2CE.
This is a general method.

13.5.2   Example: Pendulum motion

The DE describing pendulum motion is[33]

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta), \tag{6}$$

where θ gives the angle from the vertical, positive in the anti-clockwise direction, g is the gravitational acceleration constant and L the length of the pendulum.

This DE is non-linear in the dependent variable θ because of the $\sin(\theta)$ term, and cannot be solved algebraically.
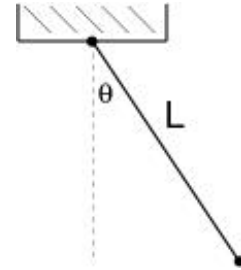
We convert our *second-order*, non-linear differential equation into a pair of coupled, *first-order*, non-linear differential equations.

Let the pendulum angular velocity be $v = \dfrac{d\theta}{dt}$. This defines a new variable $v = v(t)$.

Differentiating both sides of this equation with respect to t gives the angular acceleration

$$\frac{dv}{dt} = \frac{d^2\theta}{dt^2}. \tag{7}$$

Substitute Eq. (7) into Eq. (6). We then have the system of two **two coupled first-order** (non-linear) DEs

$$\frac{d\theta}{dt} = v, \tag{8}$$

$$\frac{dv}{dt} = -\frac{g}{L}\sin(\theta). \tag{9}$$

---

[33]This equation follows directly from Newton's Second Law for a body moving in a circular arc with $u = L\theta$, or from energy conservation.

We can now use MODEULR2/MEULR2CE on these two DEs, as described in Section 13.4.1, page 142. This program assumes coupled first-order DEs of the form

$$\frac{d\text{U}}{d\text{X}} \;=\; \text{Y}_1\,(\text{X}, \text{U}, \text{V}) \qquad \frac{d\text{V}}{d\text{X}} \;=\; \text{Y}_2\,(\text{X}, \text{U}, \text{V}).$$

We translate the DEs in the problem to **calculator variables** X, U, V. Remember that the right-hand side of the first DE, Eq. (8), is stored (in terms of X, U, V) in Y_1 on the calculator, the right-hand side of the second DE, Eq. (9), in Y_2.

Example: Plot the graph of $\theta(t)$, the solution of the pendulum equation

$$\frac{d^2\theta}{dt^2} = -10\sin(\theta)$$

or, as a first-order system,

$$\frac{d\theta}{dt} = v \qquad \frac{dv}{dt} = -10\sin(\theta),$$

on the interval $0 \le t \le 2.5$ seconds, given $\theta(0)=\pi/12\,\text{rad}$ and $v(0)=\theta'(0)=0$ (rad/s).

Find $\theta(2.5)$ and $v(2.5)$ accurate to 3 decimal places.

In practical terms, we are pulling the pendulum out to an angle of $\pi/12$ radians to the right (θ positive), and letting it go from rest ($v(0)=0$).

Step 1: Convert from problem variables to calculator variables

Comparing the pendulum equations with the calculator equations, we see that, in this case,

$$t \to \text{X} \qquad \theta \to \text{U} \qquad v = \frac{d\theta}{dt} \to \text{V}.$$

Thus,

$$\frac{d\theta}{dt} = v \;\to\; \frac{d\text{U}}{d\text{X}} = \text{V}. \qquad \text{Set } \text{Y}_1 = \text{V}.$$

$$\frac{dv}{dt} = -10\sin(\theta) \;\to\; \frac{d\text{V}}{d\text{X}} = -10\sin(\text{U}). \qquad \text{Set } \text{Y}_2 = -10\sin(\text{U}).$$

Note: $\text{Y}_1 = \text{V}$ for all second-order DEs that we solve using MODEULR2/MEULR2CE. Y_2 will be different for different second-order DEs.

Step 2: Set the ⬚window for the solution

We need to tell the program the range of times for which we want the solution. We do this by setting the window variables.

For this problem, set the ⬚window to $[0, 2.5, 1] \times [-0.5, 0.5, 0.1]$. Note that Xmin $= 0$.

Thus, we are solving the DEs for time $t=0\,\text{s}$ to time $t=2.5\,\text{s}$.

The Y range is suggested by the initial condition $\theta(0) = \pi/12 \approx 0.26$, and from experimenting. Of course you should always be in Radian mode.

Step 3: Running the program

Now run MODEULR2/MEULR2CE.

- Choose TIME PLOT.

- Set U(Xmin) $= \pi/12$ U(0) $= \theta(0) = \pi/12$.

- Set V(Xmin) $= 0$ V(0) $= v(0) = \theta'(0) = 0$.

- Start with a step size H $= 0.1$.

- Choose the graph of U(X), that is $\theta(t)$.

Step 4: Converging to the solution

The program then generates an approximate solution to the pendulum equations using the modified Euler's method. As always, we determine accuracy by reducing the step size H in powers of 10.

The table below shows the MODEULR2/MEULR2CE values for $\theta(2.5)$ and $v(2.5) = \theta'(2.5)$ for step sizes H of 0.1, 0.01 and 0.001.

The graphs for H $= 0.1$ and 0.01 in the figure below are almost indistinguishable. However, for an accuracy of 3 decimal places in the numerical values, we need the results for H $= 0.001$.
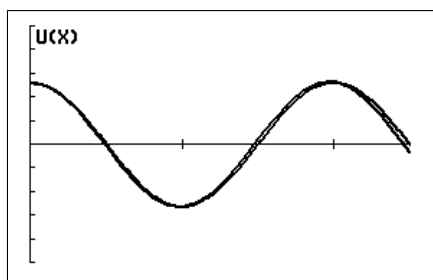
window $[0, 2.5, 1] \times [-0.5, 0.5, 0.1]$

H	$\theta(2.5)$	$v(2.5)$
0.1	-0.038	-0.842
0.01	-0.005	-0.825
0.001	-0.005	-0.825

Conclusion: On the interval $0 < t < 2.5$, a step size of 0.01 gives a good approximation to the graph of the exact answer, which looks periodic.

From the table, $\theta(2.5) = -0.005$ rad, that is slightly to the left of the equilibrium (vertical) position, and $v(2.5) = -0.825$ rad/s, that is moving to the left, **both values accurate to 3 decimal places**.

EULR2/EULER2CE does the same thing as MODEULR2/MEULR2CE using Euler's method, as does **RK2GRAPH/RK2GPHCE** using the fourth-order Runga-Kutta method.

ME2CALC/ME2CLCE and **RK2CALC/RK2CALCE** just calculate final values, and therefore run faster than the corresponding programs which plot the solutions.

13.5.3   Example: Damped oscillations

Use MODEULR2/MEULR2CE to sketch a reasonably accurate solution $\theta(t)$ to the initial-value problem for a pendulum,

$$\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} + 10\sin(\theta) = 0 \qquad \theta(0) = \pi/12 \quad \theta'(0) = 0,$$

on the interval $0 \leqslant t \leqslant 2.5$ seconds by first converting the second-order DE to two coupled first-order DEs.

What appears to be happening to the oscillations as t increases? What is different about this DE compared to the one in the previous example that might account for this?

What is the position of the pendulum (radians), and how fast and in what direction is it swinging, at $t=2.5\,\mathrm{s}$? Find values accurate to 2 decimal places.

Let

$$\frac{d\theta}{dt} = v. \tag{10}$$

Therefore,

$$\frac{d^2\theta}{dt^2} = \frac{dv}{dt}.$$

The second-order DE then becomes

$$\frac{dv}{dt} + v + 10\sin(\theta) = 0 \qquad \text{or}$$

$$\frac{dv}{dt} = -v - 10\sin(\theta). \tag{11}$$

Equations (10) and (11) are coupled first-order non-linear DEs, non-linear because of the $\sin(\theta)$.

The corresponding calculator equations are $\left(t \to \mathrm{X};\ \theta \to \mathrm{U};\ v \to \mathrm{V}\right)$

$$\frac{d\mathrm{U}}{d\mathrm{X}} = \mathrm{V} \qquad \text{and} \qquad \frac{d\mathrm{V}}{d\mathrm{X}} = -\mathrm{V} - 10\sin(\mathrm{U}).$$

Set $\mathrm{Y1} = \mathrm{V}$ and $\mathrm{Y2} = -\mathrm{V} - 10\sin(\mathrm{U})$.

Careful with the different − signs on your calculator.

The initial conditions are $\mathrm{U}(\mathrm{Xmin}) = \pi/12$ and $\mathrm{V}(\mathrm{Xmin}) = 0$.

A suitable window is $[0, 2.5, 0.5] \times [-0.3, 0.35, 0.1]$.

Run MODEULR2/MEULR2CE, TIME PLOT, with $\mathrm{H} = 0.1$, 0.01 and 0.001, from which we conclude that $\mathrm{H} = 0.1$ gives a reasonably accurate graph. The three graphs (over the page) are virtually indistinguishable.

PTO
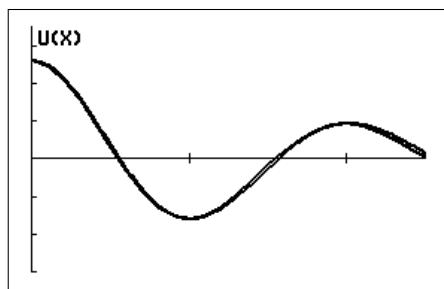
H	$\theta(2.5)$	$v(2.5)$
0.1	0.007	-0.232
0.01	0.017	-0.239
0.001	0.017	-0.239

window $[0, 2.5, 0.5] \times [-0.3, 0.35, 0.1]$

The amplitude of the oscillations is decreasing with time because the DE now contains a damping term $d\theta/dt$.

At $t = 2.5\,\text{s}$, the pendulum is $0.02\,\text{rad}$ to the right of its equilibrium position (positive θ value), swinging to the left (negative v value) at a rate of $0.24\,\text{rad/s}$; all values are accurate to 2 decimal places.

Simple harmonic motion

When θ is small, $\sin(\theta) \approx \theta$, and Eq. (6) becomes $\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{L}\theta$, a linear second-order DE with solution $\theta(t) = A\sin(\alpha\theta) + B\cos(\alpha\theta)$, where $\alpha = \sqrt{g/L}$, and A and B are arbitrary constants. This motion is called simple harmonic motion.

The spinner programs **SPSHMA/SPSHMACE** and **SPSHMB/SPSHMBCE** plot the position and the velocity versus time of a particle undergoing simple harmonic motion; parameters can be changed on-screen. SPSHMB/SPSHMBCE also plots the actual path of a particle undergoing vertical simple harmonic motion.

The program **DAMPING/DMPINGCE** plots curves of damped simple harmonic motion for different values of the damping coefficient. This shows the transition from undamped oscillations (no damping) through damped oscillations (blue) and critical damping (red) to damping with no oscillations (black).
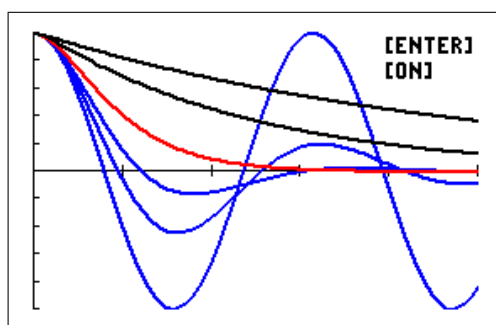
window $[0, 10, 2] \times [-1, 1, 0.2]$

Group file: Second-order DEs: DE2ORD.8xg/DE2ORDCE.8xg contains all the solvers for second-order DEs (see Section 16.2.3).

Exercises   *Solutions are in Section 13.8.*

Convert each second-order equation into an equivalent system of first-order DEs with initial conditions. Plot the solutions to each DE on $0 \leqslant x \leqslant 2$ and find $u(2)$ accurate to 4 significant digits.

1. $\dfrac{d^2u}{dt^2} + 5\dfrac{du}{dt} + 4u = 0, \quad u(0) = 7, \ u'(0) = 5.$

2. $2u'' + 2\cos(u') + u = \sin(t), \quad u(0) = 3, \ u'(0) = -1.$

13.6   Higher-order DEs

In the same way that we converted a second-order DE to two first-order DEs, we can convert a third-order DE to three first-order DEs. If the third-order DE is

$$\frac{d^3u}{dx^3} + a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = f(x),$$

let

$$v = \frac{du}{dx} \quad \text{and} \tag{12}$$

$$w = \frac{d^2u}{dx^2} \tag{13}$$

Then, the third-order DE becomes

$$\frac{dw}{dx} + aw + bv + cu = f(x). \tag{14}$$

Equations (12), (13) and (14) are a system of three coupled first-order DEs that we can solve numerically with, for example, MODEULR3/MEULR3CE. This program works the same way as the TIME PLOT option in MODEULR2/MEULR2CE but with one extra dependent variable.

In general, a DE of order n can be converted to n coupled first-order DEs, which can be solved numerically with DE20PLCE/DE20CLCE for $n \leqslant 20$ (see Section 13.4.3). The new dependent variables are the first $n-1$ derivatives of the original dependent variable.

13.7   Calculator programs for differential equations

See Section 16.6.3 for a full summary.

13.8   Solutions to exercises

Exercises   page 134

1. Sketch a slopefield for the differential equation $\dfrac{dy}{dx} = y - 2x$.

Then use it to draw the solution curve that passes through the point $(1, 0)$.

Use a window of $[-3, 3, 1] \times [-2, 2, 1]$.
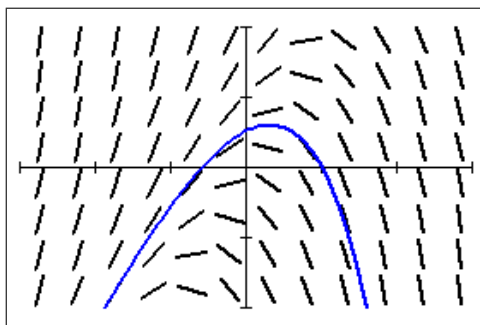
2. Sketch a slopefield for the differential equation $\dfrac{dy}{dx} = x - xy$.

Then use it to draw the solution curve that passes through the point $(1, 0)$.
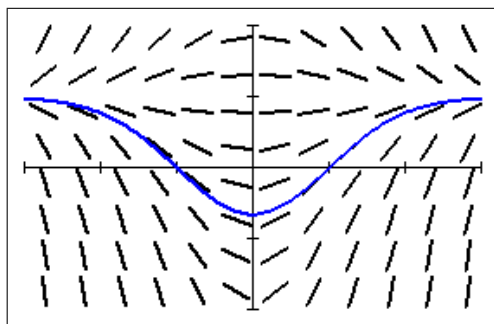
window $[-3, 3, 1] \times [-2, 2, 1]$

3. Sketch a slopefield for the differential equation $\dfrac{dy}{dx} = 2 - y$.

Then use it to draw three representative solution curves, i.e. curves showing the different types of behaviour of the solutions. Use a window $[0, 3, 1] \times [0, 4.5, 1]$.
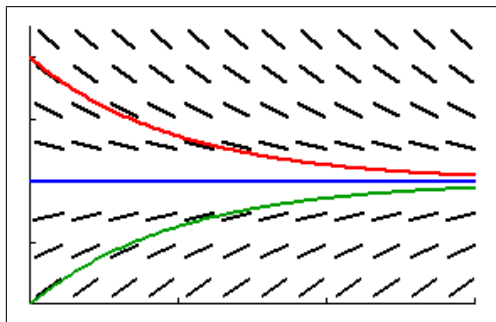
The blue horizontal line $y = 2$ is a solution of the DE.

Exercise: *Changing the step length*   page 136

Try some smaller H values, say 0.1, 0.01. The value 0.5 in the example is relatively large. What do you observe?
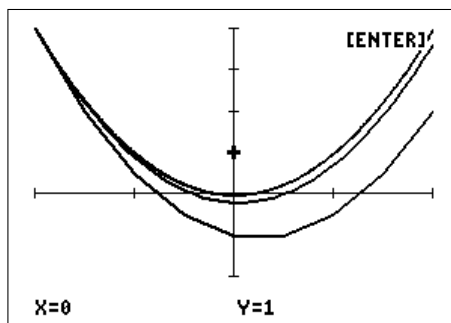
The curves appear to be converging to some final curve as H is decreased. In fact, the curve for H $=0.01$ is almost indistinguishable from the exact curve, $y=x^2$. However, the smaller H, the longer the program takes to run.

Exercise: *Changing the initial point*   page 137

Now experiment with different initial Y values. Take H $=0.1$. What do you observe?
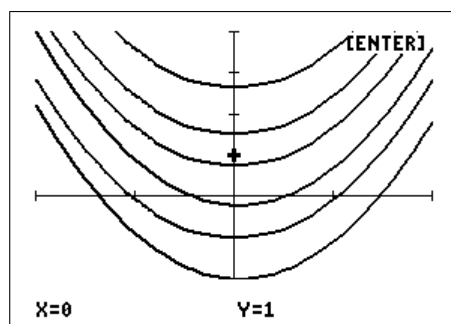
A whole series of parallel curves, one for each initial Y value.

Exercise   page 138

Use EULER1/EULER1CE to calculate $y(2)$, reducing H in powers of 10. Round values to 4 decimal places and enter them in the table.

H	$y(2)$
0.1	11.4545
0.01	11.9406
0.001	11.9940

Exercise   page 138

Use MODEULR1/MEULR1CE to complete the table below. What's your best estimate for $y(2)$ from MODEULR1/MEULR1CE? How many significant digits?

H	$y(2)$
0.1	11.9600
0.01	11.9996
0.001	12.0000

Round the last two values until they agree: $y(2) = 12.000$, accurate to 5 significant digits.

Exercises   page 140

1. Consider the differential equation $\dfrac{dy}{dx} = x - y$, with initial condition $y(0) = 1$.

   (a) Use EULER1/EULER1CE to find $y(0.4)$ *accurate* to 2 significant digits.

H	$y(0.4)$
0.1	0.7122
0.01	0.7379
0.001	0.7404

   The values in the table for $H = 0.01$ and $H = 0.001$ are both 0.74 when rounded to 2 significant digits. Therefore, accurate to 2 significant digits, $y(0.4) = 0.74$.

   (b) How do you *know* each answer in (a) is *accurate* to 2 significant digits?

   We continue reducing the step length h in powers of 10 until two successive answers are the same to the required accuracy, that is when rounded to 2 significant digits.

   We know that the approximate answers tend to the exact answer as $h \to 0$, so we can be reasonably confident that, if successive answers obtained with h reduced by a factor of 10 are the same to 2 significant digits, this is the exact answer accurate to 2 significant digits.

   (c) Use the MODEULR1/MEULR1CE program to find $y(0.4)$ *accurate* to 4 significant digits.

H	$y(0.4)$
0.1	0.7416
0.01	0.7406
0.001	0.7406

   The values in the table for $H = 0.01$ and $H = 0.001$ are both 0.7406. Therefore, accurate to 4 significant digits, $y(0.4) = 0.7406$.

(d) The exact solution to the DE for which $y(0) = 1$ is

$$y(x) \;=\; x \;-\; 1 \;+\; 2e^{-x}.$$

How does the error in using EULER1/EULER1CE to calculate $y(0.4)$ vary with H in this case? You may need to calculate $y(0.4)$ for a few more H values and keep more digits.

What return in improved accuracy does Euler's method give for the increased work caused by reducing the step size H by a factor of 10? What about the modified Euler's method?

Calculating $y(0.4)$ from EULER1/EULER1CE and MODEULR1/MEULR1CE, and the corresponding errors (error = exact − approx) gives, all rounded to 5 or 6 significant digits:

H	EULER1	Error	MODEULR1	Error
0.1	0.71220	0.02844	0.741604	-9.6×10^{-4}
0.01	0.73794	0.00270	0.740649	-8.9×10^{-6}
0.001	0.74037	0.00027	0.740640	-9.2×10^{-8}
0.0001	0.74061	0.00003	0.740640(093)	-9.3×10^{-10}

We can see that for each reduction of H by a factor of 10 (meaning the calculator has to do 10 times more calculations), we achieve a reduction in the error by a factor of about 10 using EULER1/EULER1CE, and by a factor of about 100 using MODEULR1/MEULR1CE.

For this reason, Euler's method is a 1st-order method (error reduction 10^1), whereas the modified Euler's method is a 2nd-order method (error reduction 10^2). The standard method in many calculations is a 4th-order Runge-Kutta method (error reduction 10^4; Section 13.3.3)

2. (a) Find $P(3)$ and $P(12)$, accurate to 3 significant digits, if $P(0) = 1$ and

$$\frac{dP}{dt} \;=\; 0.3e^{-0.1t^2}P.$$

In calculator variables, the DE is

$$\frac{dY}{dX} = 0.3e^{-0.1X^2}Y.$$

Set $Y_1 = 0.3e^{\wedge}(-0.1X^2)Y$, windows of $[0, 3, 1] \times [-0.3, 3, 1]$ and $[0, 12, 2] \times [-0.3, 3, 1]$, respectively. Run MODEULR1/MEULR1CE with X0 = 0, Y0 = 1 and reducing step lengths H. The table below shows the results.

H	$P(3)$	$P(12)$
1	1.97	2.30
0.1	1.99	2.32
0.01	1.99	2.32

Therefore, accurate to 3 significant digits, $P(3) = 1.99$ and $P(12) = 2.32$.

(b) What is the approximate value of $P(t)$ as $t \to \infty$?

The graph of $P(t)$ looks to have levelled out at $P \approx 2.32$: this is the approximate limiting value as $t \to \infty$.

3. (a) Use Euler's method to approximate $y(2)$ if $\dfrac{dy}{dx} = \dfrac{1}{x}$ and $y(1) = 0$ accurate to 3 significant digits.

H	$y(2)$
0.1	0.7188
0.01	0.6956
0.001	0.6934
0.0001	0.6932

Therefore, $y(2) = 0.693$, accurate to 3 significant digits.

(b) What step length do you need in MODEULR1/MEULR1CE to obtain an answer accurate to 5 significant digits?

H	$y(2)$
0.1	0.69377
0.01	0.69315
0.001	0.69315

A step length of 0.01 gives an answer accurate to 5 significant digits, verified by using a step length of 0.001.

4. (a) Find $y(1.5)$ accurate to 3 significant digits for the initial-value problem

$$y' + \tan(x)\, y = \cos^2(x) \qquad y(0) = -1$$

using MODEULR1/MEULR1CE. In calculator variables, the DE is

$$\frac{dY}{dX} = -\tan(X)Y + \cos(X)^2.$$

Set $Y_1 = -\tan(X)Y + \cos(X)^2$ (radians) and a window of $[0, 1.5, 0.5] \times [-1, 0.15, 0.2]$. Run MODEULR1/MEULR1CE with $X0 = 0$, $Y0 = -1$ and reducing H.

H	$y(1.5)$
0.1	-2.37×10^{-3}
0.01	-1.88×10^{-4}
0.001	-1.77×10^{-4}
0.0001	-1.77×10^{-4}

Therefore, accurate to 3 significant digits, $y(1.5) = -1.77 \times 10^{-4}$.

(b) The solution to the initial-value problem above is $y(x) = \sin(x)\cos(x) - \cos(x)$ (on the interval $0 < x < \pi/2$). Use this to check that your answer to (a) is indeed accurate to 3 significant digits.

From the exact solution,

$$y(1.5) = \sin(1.5)\cos(1.5) - \cos(1.5) = -1.77 \times 10^{-4},$$

rounded to 3 significant digits, in agreement with MODEULR1/MEULR1CE.

Exercises page 141

Repeat the previous exercises using the fourth-order Runge-Kutta method instead of the modified Euler's method.

1. Consider the differential equation $\dfrac{dy}{dx} = x - y$, with initial condition $y(0) = 1$.

    (c) Use the RK1CALC/RK1CALCE program to find $y(0.4)$ *accurate* to 4 significant digits.

H	$y(0.4)$
0.1	0.7406
0.01	0.7406

    The values in the table for H $=0.1$ and H $=0.01$ are both 0.7406. Therefore, accurate to 4 significant digits, $y(0.4) = 0.7406$.

    (d) The exact solution to the DE for which $y(0) = 1$ is

$$y(x) \;=\; x \;-\; 1 \;+\; 2e^{-x}.$$

    How does the error in using RK1CALC/RK1CALCE to calculate $y(0.4)$ vary with H in this case? You may need to calculate $y(0.4)$ for a few more H values and keep more digits.

    What return in improved accuracy does the fourth-order Runge-Kutta method give for the increased work caused by reducing the step size H by a factor of 10?

    Calculating $y(0.4)$ from RK1CALC/RK1CALCE, and the corresponding errors gives (error $=$ exact $-$ approx),

H	RK1CALCE	Error
0.1	0.7406405778	-4.87×10^{-7}
0.01	0.7406400921	-2.87×10^{-11}
0.001	0.7406400921	$< 10^{-14}$

    We can see that for each reduction of H by a factor of 10 (meaning the calculator has to do 10 times more calculations), we achieve a reduction in the error using RK1CALC/RK1CALCE of about a factor of 10^4.

PTO

2. (a) Find $P(3)$ and $P(12)$, accurate to 3 significant digits, if $P(0)=1$ and

$$\frac{dP}{dt} = 0.3e^{-0.1t^2}P.$$

H	$P(3)$	$P(12)$
1	1.993	2.318
0.1	1.993	2.318

Therefore, accurate to 3 significant digits, $P(3)=1.99$ and $P(12)=2.32$.

(b) What is the approximate value of $P(t)$ as $t \to \infty$?

The graph of $P(t)$ looks to have levelled out at $P \approx 2.32$: this is the approximate limiting value as $t \to \infty$.

3. (b) What step length H do you need in RK4CALC/RK4CALCE to obtain $y(2)$ if $\dfrac{dy}{dx} = \dfrac{1}{x}$ and $y(1)=0$, accurate to 5 significant digits?

H	$y(2)$
0.1	0.69315
0.01	0.69315

A step length of 0.1 gives an answer accurate to 5 significant digits, verified by using a step length of 0.01.

4. (a) Find $y(1.5)$ accurate to 3 significant digits for the initial-value problem

$$y' + \tan(x)\,y = \cos^2(x) \qquad y(0) = -1$$

using RK4CALC/RK4CALCE.

H	$y(1.5)$
0.1	-2.52×10^{-4}
0.01	-1.77×10^{-4}
0.001	-1.77×10^{-4}

Therefore, accurate to 3 significant digits, $y(1.5) = -1.77 \times 10^{-4}$.

(b) The solution to the initial-value problem above is $y(x) = \sin(x)\cos(x) - \cos(x)$ (on the interval $0 < x < \pi/2$). Use this to check that your answer to (a) is indeed accurate to 3 significant digits.

From the exact solution,

$$y(1.5) = \sin(1.5)\cos(1.5) - \cos(1.5) = -1.77 \times 10^{-4},$$

rounded to 3 significant digits, in agreement with RK4CALC/RK4CALCE.

Exercises page 145

1. Plot the solutions to the following system on $0 \leqslant x \leqslant 2$, with initial conditions $y_1(0) = 8$ and $y_2(0) = 2$:

$$\frac{dy_1}{dx} = -2y_1 + 3y_2 + 12e^x$$

$$\frac{dy_2}{dx} = y_1 - 4y_2.$$

Find $y_1(2)$ and $y_2(2)$ accurate to 4 significant digits.

Here, $x \to$ X; $y_1 \to$ U; $y_2 \to$ V. The equations in calculator variables are then

$$\frac{d\text{U}}{d\text{X}} = -2\text{U} + 3\text{V} + 12e\wedge(\text{X})$$

$$\frac{d\text{V}}{d\text{X}} = \text{U} - 4\text{V}.$$

The initial conditions are $\text{U}(0) = 8$, $\text{V}(0) = 2$.

Running any of EULER2, MODEULR2 or RK2GRAPH, or the CE equivalents gives the following graphs of $y_1(t)$ (top) and $y_2(t)$.
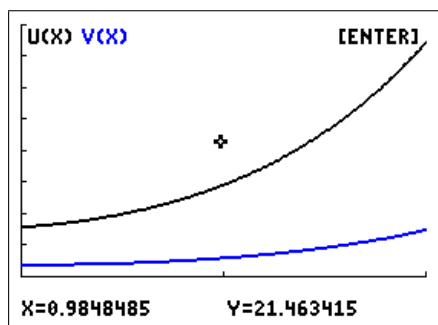
window $[0, 2, 1] \times [0, 40, 5]$

The values for $y_1(2)$ are given below.

H	EULER2	MODEULR2	RK2GRAPH
0.1	36.454	37.420	37.351
0.01	37.262	37.351	37.351
0.001	37.342	37.351	–
0.0001	37.350	–	–

More EULER2 calculations required (probably overnight).

Therefore, $y_1(2) = 37.35$, accurate to 4 significant digits. From the same calculations, $y_2(2) = 7.524$, accurate to 4 significant digits.

2. Let $u_P(t)$ be the amount (mass) of a drug in the blood plasma at time t days, and let $u_T(t)$ be the amount in the tissue (organs in the body) for certain positive rate constants k_a, k_b and k_e. The following equations model the process in which the drug enters the plasma at rate $g(t)$, moves between the plasma and the tissue, and is excreted from the plasma:

$$\frac{du_P}{dt} = -k_b u_P - k_e u_P + k_a u_T + g(t)$$

$$\frac{du_T}{dt} = k_b u_P - k_a u_T.$$

Consider the case in which $k_a = 4$, $k_b = 2$, $k_e = 3$ and $g(t) = 0$, giving equations

$$\frac{du_P}{dt} = -5u_P + 4u_T$$

$$\frac{du_T}{dt} = 2u_P - 4u_T.$$

Suppose the initial conditions are $u_P(0) = 1$ and $u_T(0) = 0$: a dose of 1 unit of a drug is injected into the plasma or bloodstream at time $t = 0$, with none in the tissue. Plot $u_P(t)$ and $u_T(t)$ on the interval $0 \leqslant t \leqslant 2$ and comment: do these graphs make sense for the amount of drug in the plasma and tissue?

Here, $t \to$ X; $u_P \to$ U; $u_T \to$ V. The equations in calculator variables are then

$$\frac{d\text{U}}{d\text{X}} = -5\text{U} + 4\text{V}$$

$$\frac{d\text{V}}{d\text{X}} = 2\text{U} - 4\text{V}.$$

The initial conditions are U$(0) = 1$, V$(0) = 0$.

Running any of EULER2, MODEULR2 or RK2GRAPH, or the CE equivalents gives the following graphs of $u_P(t)$ (top) and $u_T(t)$.
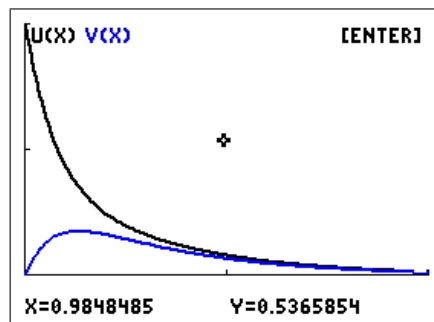
window $[0, 2, 1] \times [0, 1, 0.5]$

The graphs make sense. The intial drug in the plasma is lost to the tissue or excreted. Although some comes back from the tissue, it is always less than the amount lost, so the amount of the drug in the plasma always decreases.

The drug in the tissue increases from zero as it receives drug from the plasma. However, this amount decreases as the drug is excreted, and eventually the amount of drug in the tissue reaches a maximum, then decreases.

Exercises page 155

Convert each second-order equation into an equivalent system of first-order DEs with initial conditions. Plot the solutions to each DE on $0 \leqslant x \leqslant 2$ and find $u(2)$ accurate to 4 significant digits.

1. $\dfrac{d^2u}{dt^2} + 5\dfrac{du}{dt} + 4u = 0, \quad u(0) = 7, \ u'(0) = 5.$

Let $\dfrac{du}{dt} = v$. Then, the second-order DE becomes

$$\frac{dv}{dt} + 5v + 4u = 0, \quad u(0) = 7, \ v(0) = 5,$$

or

$$\frac{dv}{dt} = -5v - 4u \quad u(0) = 7, \ v(0) = 5.$$

Here, $t \to$ X; $u \to$ U; $v \to$ V. The equations in calculator variables are then

$$\frac{d\mathrm{U}}{d\mathrm{X}} = \mathrm{V}$$

$$\frac{d\mathrm{V}}{d\mathrm{X}} = -5\mathrm{U} - 4\mathrm{V}.$$

The initial conditions are U$(0) = 7$, V$(0) = 5$.

Running any of EULER2, MODEULR2 or RK2GRAPH, or the CE equivalents gives the following graph of $u(t)$.
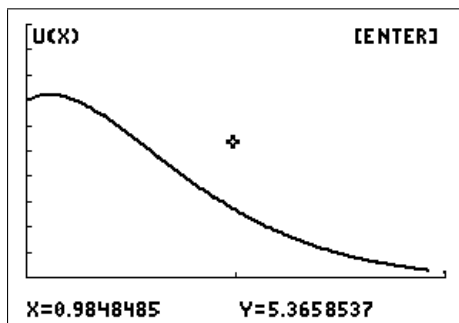
window $[0, 2, 1] \times [0, 10, 1]$

The values for $u(2)$ are given below.

H	EULER2	MODEULR2	RK2GRAPH
0.1	0.08094	0.27534	0.26308
0.01	0.24473	0.26319	0.26308
0.001	0.26124	0.26308	–
0.0001	0.26290	0.26308	–

More EULER2 calculations required (probably overnight).

Therefore, $u(2) = 0.2631$, accurate to 4 significant digits.

2. $2u'' + 2\cos(u') + u = \sin(t), \quad u(0) = 3, \; u'(0) = -1.$

Let $\dfrac{du}{dt} = v.$ Then, the second-order DE becomes

$$2\frac{dv}{dt} + 2\cos(v) + u = \sin(t), \quad u(0) = 3, \; v(0) = -1,$$

or

$$\frac{dv}{dt} = -\cos(v) - 0.5u + 0.5\sin(t) \quad u(0) = 3, \; v(0) = -1.$$

Here, $t \to$ X; $u \to$ U; $v \to$ V. The equations in calculator variables are then

$$\frac{dU}{dX} = V$$

$$\frac{dV}{dX} = -0.5U - \cos(V) + 0.5\sin(X).$$

The initial conditions are U$(0) = 3$, V$(0) = -1$.

Running any of EULER2, MODEULR2 or RK2GRAPH, or the CE equivalents gives the following graph of $u(t)$.
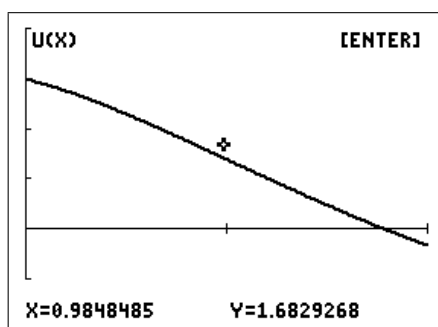
window $[0, 2, 1] \times [-1, 4, 1]$

The values for $u(2)$ are given below.

H	EULER2	MODEULR2	RK2GRAPH
0.1	-0.39941	-0.32823	-0.32714
0.01	-0.33417	-0.32714	-0.32714
0.001	-0.32784	-0.32714	$-$
0.0001	-0.32721	$-$	$-$

More EULER2 calculations required (probably overnight).

Therefore, $u(2) = -0.3271$, accurate to 4 significant digits.

14 Population Modelling 2
Logistic and Epidemic Models

14.1 Logistic model

The logistic model arose from an attempt by Verhulst to come up with a more realistic population model than Malthus' exponential model (see Population Modelling 1 in Volume 1 of this book). He reasoned that no organism grows without bound, otherwise the Earth would be covered in this organism. Restrictions to growth are imposed by the need for food and space, so that the growth rate k, assumed constant in the exponential model, must decrease as the population increases. Verhulst chose the simplest such form for k, a linear decrease with population P: $k = a - bP$, where a and b are constants. This led to the so-called logistic curves (see Section 14.1.1 below), which start out like exponentials but eventually saturate or tend to a constant value.

14.1.1 Mathematical background

This section briefly outlines the steps to the logistic model in terms of differential equations. You can skip down to *Solution* if you haven't covered/encountered differential equations.

The assumption that the (instantaneous) rate of change (increase or decrease) in a population $P(t)$ at time t is proportional to the population at that time gives rise to the exponential differential equation

$$\frac{dP}{dt} = kP,$$

where the constant k is the growth (or decay) rate. Solutions are of the form $P(t) = P_0 e^{kt}$, where $P_0 = P(0)$ is the initial population.

Putting the Verhulst growth rate $k = a - bP$ into the exponential differential equation above, re-arranging and renaming some constants gives the logistic differential equation

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{K} \right),$$

where it turns out that K is the maximum sustainable population or carrying capacity.

Solution: The solution to the logistic differential equation is

$$P(t) = \frac{KEe^{kt}}{Ee^{kt} - 1}, \tag{1}$$

where $E = P_0/(P_0 - K)$, with $P_0 = P(0)$ the initial population ($P_0 \neq K$).

If $P_0 = K$, the solution is $P(t) = K$, a constant or equilibrium solution. $P(t) = 0$ is the other equilibrium solution but not of great interest in population modelling other than acknowledging that it makes sense. Some logistic curves $P(t)$ are shown below.
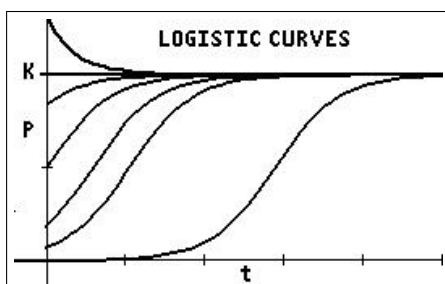
Logistic curves $P(t)$ are close to exponential for small t but, instead of increasing indefinitely like the exponentials, they level out and approach the constant value K asymptotically. Curves starting above the carrying capacity K decrease down to it asymptotically.

14.1.2 Logistic problem

Exercise: A population $P(t)$ of fish in a pond is modelled by Eq. (1) with $k=2.5$ and carrying capacity $K=100$. Time t is measured in weeks. Initially there are 10 fish in the pond.

1. Write down the formula for $P(t)$ and graph it for $0<t<5$.

2. How many fish are there after 1 week?

3. How long does it take the fish population to reach 80?

4. How many fish are there after a long time?

Solutions to this problem are in Section 14.4.1.

14.2 Discrete logistic model

Here we use a discrete version of the Verhulst or logistic model, called the discrete logistic model, to predict the growth of a population of bacteria or kangaroos; in a discrete model, time goes in steps rather than being continuous, as it was in the previous model. The model is in the form of a difference equation which tells you how to calculate the population after the next time step, P_{n+1}, if you know the population P_n now:

$$P_{n+1} = AP_n(1-P_n), \tag{2}$$

where A is a constant

14.2.1 Exercise: Bacteria

In this model, P_n is a measure of the population (in millions of bacteria) at the end of the nth hour and A is a number that depends on how fast the bacteria reproduce. For our calculations, we take $A=2$ and the starting population $P_0=0.1$.

To calculate P_1, the population after 1 hour, put $n=0$ and $P_0=0.1$ in Eq. (2):

$$P_1 = 2P_0(1-P_0) = 2 \times 0.1 \times (1-0.1) = 0.18.$$

To calculate P_2, the population after the second hour, put $n=1$ in Eq. (2):

$$P_2 = 2P_1(1-P_1) = 2 \times 0.18 \times (1-0.18) = 0.2952,$$

and so on. After a few more steps (hours), you should find the population stabilises at a particular number. *What is the number?*

Calculator hint

To speed up this process, on a calculator, type:

$0.1 \rightarrow$ | alpha | | P | | enter | store the starting population in memory P;

$2P(1\text{–}P) \rightarrow$ | P | | enter | evaluate Eq. (2) and store the result back in P.

This will give you the next value for P. If you now keep pressing | enter |, the calculator will repeatedly execute the last line to give successive values for P.

Next let $A = 3.2$ and keep $P_0 = 0.1$:

    $0.1 \rightarrow$ | alpha | | P | | enter |;

    $3.2\text{P}(1\text{–P}) \rightarrow$ | P | | enter |;

and keep pressing | enter |.

You'll need to run the population for about 18 (model) hours (time steps) this time before it settles down. What happens here? *Draw a plot of population versus time.*

Now try $A = 3.8$ and $P_0 = 0.1$. This one is weird! The population varies wildly between 0.1 and 1, with no hope of prediction. *Plot this one too.* You've discovered chaos (the mathematical version). *What happens with other values of A and P_0?*

Solutions to the above are in Section 14.4.2.

Sequence grapher: can be used to graph values of P_n vs n (see Section 14.3.7 for details). The LOGISTIC/LGSTCE/SPLOGIST/SPLGSTCE program (Section 14.2.3) sets this up for you for the bacteria here and for the kangaroos in the next section.

14.2.2   Exercise: Kangaroo management

Part of an UNSW Canberra Maths Lab adapted from *Stimulating Mathematical Interest with Dynamical Systems* by M.B. Durkin, The Maths Teacher 89, 242–24 (1996).

You are hired by the State Forestry Department, with your main task to assist in the management of the kangaroo population in a remote forest called Hamt Reserve. The possibility of culling of kangaroos in the reserve is under consideration.

The kangaroo population in the reserve is given by the discrete logistic model, a difference equation,
$$P_{n+1} = 1.8P_n - 0.8(P_n)^2, \tag{3}$$

where P_n is the number of kangaroos in the reserve at the end of year n in tens of thousands, i.e. *one unit of P equals 10,000 kangaroos.* At the end of 2005, there were 8000 kangaroos in the reserve ($P_0 = 0.8$).

The first task

As a training exercise, management asks you to model and report on a scenario containing several events that would affect the kangaroo population.

Write a short report on the outcome of the following scenario. The report should include a mathematical analysis with calculations, tables and/or graphs to substantiate your conclusions.

The scenario

- If there are no natural disasters in 2006, what is the kangaroo population at the end of 2006? Do this and the following calculations manually (without a program) using Eq. (3).[34]

- Unfortunately, at the end of 2006, there is a short but fatal outbreak of the dreaded rootoxis which kills around 4000 kangaroos. What is the population of kangaroos at the end of 2007? When does the kangaroo population recover to more than 9000 kangaroos if are no more natural disasters?

---

[34] *Calculator hint*: As in Section 14.2.1, store the initial population in memory P and repeatedly execute the calculation $1.8\text{P} - 0.8\text{P}^2 \rightarrow \text{P}$ by pressing | enter | the required number of times.

- Following the rootoxis epidemic, on Christmas Day 2008 there is a forest fire in a nearby forest which results in 2000 kangaroos from that forest migrating into Hamt Reserve. What is the population of kangaroos in Hamt Reserve at the end of 2009?

- After these two events, there are no more natural disasters. What is the kangaroo population after a long time? The number here is the limiting capacity or maximum sustainable population of the reserve.

Effect of culling

Impressed by your previous report, management has now put you in charge of undertaking a feasibility study into whether culling of kangaroos is necessary/desirable in Hamt Reserve. Your analysis will be a crucial factor in the decision-making process.

Write a report addressing the following questions. Again, a mathematical analysis including calculations, tables and/or graphs is required to substantiate your conclusions. Add an executive summary for your boss, summarising your findings and making suitable recommendations.

1. What is the modified form of Eq. (3) if H kangaroo units are culled each year?

   We assume here, for simplicity, that all the kangaroos are killed close to the end of the year, otherwise the killing of the female kangaroos in particular would affect the number of births and deaths, and consequently the growth rate.

2. What would happen if 720 kangaroos were culled each year ($H = 0.072$), a value used in a nearby reserve? Assume the initial population is that given above for the year 2005, $P_0 = 0.8$. What is the long-term population?

   What if the initial population were $P_0 = 0.3$? $P_0 = 0.095$?

3. What would happen if 2400 kangaroos were culled each year ($H = 0.24$)? Assume again that $P_0 = 0.8$. What is the long-term population?

   What if the initial population were $P_0 = 1$? $P_0 = 1.5$?

4. What about $H = 0.2$? It turns out[35] that this is the largest number of kangaroos which could be culled annually without the kangaroos dying out in Hamt Reserve. Note that the initial population must be larger than 0.5. What is the long-term population in this case?

Solutions to the above are in Section 14.4.2.

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[35]Experiment and see — the LOGISTIC/LGSTCE program, next section, might help.

14.2.3   Logistic-model programs

Available at *www.XXX.*

Sets up the graphics for a population of bacteria obeying the discrete logistic equation (Section 14.2.1) or for a population of kangaroos obeying the discrete logistic equation with culling (Section 14.2.2).

LOGISTIC/LGSTCE

Use: Run the program. Select BACTERIA or KANGAROOS. Input the appropriate parameters at the prompts ($0 < u(0) < 1$ for the bacteria). The program plots population versus cycle number (time). Use the arrow keys to trace the graph or press ⎡enter⎤ to return to the main menu.

When the program has finished, choose QUIT from the main menu. Here you can either keep the equations for manual plotting (e.g. with a different ⎡window⎤: Option 1) or you can have the equations and other settings deleted (Option 2). If you choose Option 1, when you have finished rerun the program, QUIT and select Option 2 to tidy up.

SPLOGIST/SPLGSTCE allows you to change the relevant parameters on-screen and see the result immediately replotted.

14.3   SIR epidemic model

14.3.1   Introduction to epidemic modelling

Based on www.stat.columbia.edu/∼regina/research/notes123.pdf.

The modelling of infectious diseases is a tool that is used to study the mechanisms by which diseases spread, to predict the future course of an outbreak and to evaluate strategies to control an epidemic.

Infectious agents have had decisive influences on the history of mankind. Fourteenth-century Black Death took the lives of about a third of Europe's population at the time. Thucydides described the Plague of Athens ($430 - 428$ BC): 1,050 of 4,000 soldiers on an expedition died of a disease. He gave a detailed account of symptoms, some so horrendous that the last one – amnesia – seems a blessing (Bailey, 1975). An interesting feature of this account is that there was no mention of person-to-person contagion, which we now suspect with most new diseases; it was not until the 19th century that this was beginning to be discussed.

The practical use of epidemic models relies heavily on the assumptions underlying the models. A reasonable model does not have to include all possible effects but should incorporate the main mechanisms that influence disease propagation in the simplest possible fashion. Great care should be taken before epidemic models are used for prediction of real phenomena. However, even simple models should, and often do, pose important questions about the underlying mechanisms of infection spread and possible means of control of the disease or epidemic.

The classical papers by Kermack and McKendrick (1927, 1932, 1933) have had a major influence on the development of mathematical models for disease spread, and are still relevant in many epidemic situations. The first of these papers laid out a foundation for modelling infections which confer complete immunity after recovery (or, in the case of lethal diseases, death). If infected individuals are introduced into a large population, a basic problem is to describe the spread of the infection within the population as a function of time. One of the

most important questions is whether the epidemic comes to an end only when all the initially susceptible individuals have contracted the disease or if some interplay of infectivity, recovery and mortality factors may result in epidemic "die out", with many susceptibles still present in the unaffected population.

In their first paper, Kermack and McKendrick started with the assumption that all members of the community were initially equally susceptible to the disease, and that complete immunity was conferred after the infection. The population was divided into three distinct classes: susceptibles S, healthy individuals who can catch the disease; infecteds I, those who have the disease and can transmit it; and removed R, individuals who have had the disease and are now immune to the infection (or removed from further propagation of the disease by some other means). Schematically, the individual goes through consecutive states S→I→R; such models are often called SIR models.

Bailey, NTJ (1975). *The Mathematical Theory of Infectious Diseases and its Applications.* 2nd ed., Hafner Press, New York.

Ethier, SN and Kurtz, TG (1986). *Markov Processes. Characterization and Convergence.* Wiley, New York.

Kermack, WO and McKendrick, AG (1927). Contributions to the mathematical theory of epidemics, i. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* **115**, 700–721.

Kermack, WO and McKendrick, AG (1932). Contributions to the mathematical theory of epidemics ii — the problem of endemicity. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* **138**, 55–83.

Kermack, WO and McKendrick, AG (1933). Contributions to the mathematical theory of epidemics iii — further studies of the problem of endemicity. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* **141**, 94–122.

14.3.2   The problem

In Section 14.3.3, differential equations (continuous-time models) are used to model the spread of an epidemic through a population; the model here is the basic SIR model. Several approaches are used to obtain information and to make predictions: numerical solutions are obtained with a TI-84/CE graphics calculator; analytical and graphical techniques extract more information from the equations. The continuous SIR model is then applied to an epidemic scenario.

In Section 14.3.5, difference equations (discrete-time models) are used to model the spread of an epidemic. The discrete SIR model is given and applied to the same epidemic scenario.

Your report

Compile your answers to the problems below, together with relevant plots[36] and any other material, into a coherent report that can be understood by one of your Maths colleagues.

PTO

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[36]You can print out graphics-calculator screens or save them as .png files using the calculator-to-computer cable and TI-Connect/TI-Connect CE software.

14.3.3   Continuous SIR model

Numerical approach

In the SIR model, the population is divided into three categories, **susceptible**, **infected** and **removed**, with $S(t)$, $I(t)$ and $R(t)$ being the respective numbers in each at time t days. Susceptible persons can catch the disease; infected persons are infectious and can therefore transmit the disease to susceptibles; removed persons have recovered, are then assumed immune to the disease and cannot spread it.

Let t be measured in days. The differential equations modelling this situation are

$$\frac{dS}{dt} \;=\; -\beta SI \tag{1}$$

$$\frac{dI}{dt} \;=\; \beta SI - \alpha I \tag{2}$$

$$\frac{dR}{dt} \;=\; \alpha I \tag{3}$$

where α (the recovery rate) and β (the transmission rate) are positive constants. S, I and R cannot be negative.

βSI in Eq. (1) is the number of new infections per day. These people move to the infected category, thereby decreasing S in Eq. (1) and increasing I in Eq. (2).

βSI models the transmission of the disease. The expression here is one of several possible choices. It is called the *Law of Mass Action*, which states that the rate of new cases is proportional to the product of S and I. Transmission occurs when an infected meets a susceptible, so intuitively, transmission should depend somehow on the numbers of each. In this form, the number of contacts each individual makes per day is assumed to be large in a large population, and small in a small population.

People recover from the disease according to the second term on the RHS of Eq. (2), and move to the removed category. Assuming all in R are alive, no equation term represents births or deaths in this simplified SIR model. Deaths from the disease could be included in the removed category if necessary. Infectious diseases such as influenza spread rapidly, then 'burn out' over a period of weeks or months. Natural births and deaths in the population can therefore be ignored on this timescale.

The SIR equations cannot be solved algebraically, so we look to numerical solutions. To make the problem ready for a numerical solution, we need to specify values for the constants α and β, and for the initial conditions $S(0)$, $I(0)$ and $R(0)$. For our calculations, we assume $R(0)=0$, so that $S(0)+I(0)=N$, the total population, which is constant (see Question 6).

We assume the epidemic is over once I has dropped below 1.

The next few exercises aim to help us gain some intuition into the model. They use the SIR/SIRCE program, which solves the SIR system of differential equations numerically using the modified Euler's method.

Solutions are given in Section 14.4.3.

1. Read the instructions for the SIR/SIRCE program in Appendix 1.

Run SIR/SIRCE and choose TIME PLOT.

Assume a total population $N = 1000$.

Run the model for 50 days (Tmax).

Set $\alpha = 0.2$ and $\beta = 0.0008$.

As initial conditions, set $S(0) = 999$, so that $I(0) = 1$ (you need at least one infective). This models introducing the disease into essentially a fully susceptible population.

Start with a step length H of 1 day.

Check that you get $S(50) = 20.8$ and $I(50) = 1.0$, all rounded to 1 decimal place.

Then, $R(50) = 1000 - S(50) - I(50) = 978.3$.

SIR/SIRCE graphs S, I and R as functions of time on the one plot.

What do the curves tell you about this epidemic?

2. Decreasing the step length H increases the accuracy of the results but the calculations take longer. Decrease H by a factor of 10 to check the accuracy of your results in Q1; running the program with just one step length does not tell you how accurate your results are.

Give values for $S(50)$, $I(50)$ and $R(50)$, rounded to an appropriate number of decimal places, and **accurate** to this number of decimal places.

3. The **size** of an epidemic is the total number of people infected during the epidemic.

What was it for the case above (after 50 days)?

How many people out of the population of 1000 remained free of the disease?

Hint: The numbers displayed after the plot will help you here.

Check the accuracy of your results by:

- running the model for say 75 days to see if the values for S and I change significantly after 50 days;
- decreasing the step length to check the accuracy of the results after 75 days.

4. Run SIR/SIRCE, keeping $\alpha = 0.2$ fixed and varying β.

Try some values of β centred on 0.0008, such that there are significant changes in the graphs.

Describe how the curves change for different values of β, particularly with regard to the duration of the epidemic, the time to peak I and the maximum value of I.

Remembering that β determines the rate of transmission of the disease, explain the changes.

5. Run SIR/SIRCE, keeping $\beta = 0.0008$ fixed and varying α.

Try some values of α centred on 0.2, such that there are significant changes in the graphs.

α is the reciprocal of the mean infective period, the mean time that a person who has caught the disease remains infectious.

Describe how the curves change for different values of α, particularly with regard to the duration of the epidemic, the time to peak I and the maximum value of I.

Remembering that α determines how long, on average, a person is infectious, explain the changes.

Algebraic approach

Section 14.3.3 showed that it is relatively easy to get approximate solution curves for an epidemic, and to understand how changing the parameter values affects the solution. Although we cannot solve Eqs. (1), (2) and (3) algebraically, we can find other algebraic ways of obtaining information about the epidemic, and identify some important summary quantities. This we do in this section.

You should do the questions here for general values of the parameters α, β and N, and general initial values $S(0)$ and $I(0)$. But by all means use numerical values to check your results.

6. As there are no deaths from the disease in our model, and natural births and deaths are ignored, we expect that the total population, $N = S + I + R$, will not change.

Show that, according to the model, Eqs. (1), (2) and (3), N really is a constant.
Hint: If N is a constant, what do we know about dN/dt?

Explain then why solving three DEs is then unnecessary (solving two will be sufficient).

7. The rising phase of an epidemic (as seen in an $I(t)$ curve) is a worrying time, which lasts until the epidemic 'peaks' and enters a waning stage.

Show from Eq. (2) that the number of cases (the number of persons who have caught the disease: $S(0) - S$) up to the time of peak I can be predicted in advance, if reliable estimates of α and β are available. *Hint*: Think max/min I values.

Compare this theoretical value with the approximate value from your graph in Q2.

8. According to Eq. (1), the number of susceptibles never increases. Use Eq. (2) and your thinking from Question 7 to decide what happens if the disease is introduced into a population which has $S(0) < \alpha/\beta$. What if $S(0) > \alpha/\beta$?
Hint: Think about dI/dt at $t = 0$ in the case when there is/isn't an epidemic.

The **basic reproductive number** in the SIR model is $R_0 = \beta S(0)/\alpha$. What is the threshold value of R_0 for an epidemic?

PTO

Graphical approach: Phase plots

9. Now use PHASE PLOT in SIR/SIRCE — this generates an (approximate) graph of I versus S, called a *phase plot* or *phase portrait*.

Take $\alpha = 0.2$, $\beta = 0.0008$, $N = 1000$ (as in Questions 1–3) and several different sets of initial conditions. Start with $S(0) = 999$ ($I(0) = 1$). Then take $S(0) = 900$ and $S(0) = 800$ using Option 3 in the NEXT menu. Choose the number of points (time steps) so that the epidemic is over ($I < 1$) by the final point.

You may need to change the $\boxed{\text{window}}$ (*from the option in the program*) so that the graph fills the screen. Don't forget to check for accuracy.

Print out the plot. Label points that are significant in the course of the epidemic, such as the initial point, the point of maximum I and the point at which the epidemic is taken as finished. Give the value of R_0 for each curve.

Describe what these phase plots tell you about the possible epidemics with these values of α, β and N.

10. As $S(t)$ always decreases as t increases, S can be used as a time-like variable.

Use the chain rule[37] to show from Eqs. (1) and (2) that

$$\frac{dI}{dS} = -1 + \frac{\alpha}{\beta}\frac{1}{S}.$$

Solve this DE[38] to get

$$I(t) = -S(t) + \frac{\alpha}{\beta}\ln\big(S(t)\big) + D, \tag{4}$$

where D is an arbitrary constant. In this case therefore, we can find an exact expression for the phase portrait.

Choose $\alpha = 0.2$, $\beta = 0.0008$, and use your calculator or other graphics to plot I as a function of S $\big($as a Y(X) graph$\big)$ for some different values of D.

What is a sensible $\boxed{\text{window}}$*?*

What determines the value of D in any given situation?

What value of D corresponds to the graph you generated in Question 9?

Do the two graphs then give the same values?

For most systems of DEs, we cannot find an exact expression like Eq. (4), and must therefore use a numerical approach, like we did in Question 9.

PTO

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[37] $\frac{dI}{dS} = \frac{dI}{dt}\frac{dt}{dS} = \frac{dI}{dt}\big/\frac{dS}{dt}$.

[38] or show that the given solution <u>is</u> a solution by substituting it into the DE and showing LHS $=$ RHS

11. A special case of Eq. (4) is the **final-size equation**

$$I_\infty - I_0 = S_0 - S_\infty + \frac{\alpha}{\beta} \ln\left(\frac{S_\infty}{S_0}\right), \tag{5}$$

where $S_0 = S(0)$, $I_0 = I(0)$, and I_∞ and S_∞ are the respective limits as $t \to \infty$.

Explain how this equation follows from Eq. (4).

It is usual to assume that $I_0 = I_\infty = 0$ (absence of the disease before and after the epidemic).

In Question 3 you found the final size of an epidemic for $\alpha = 0.2$, $\beta = 0.0008$ and $N = 1000$. Here, predict the final size with these values using Eq. (5). Compare the two results.

Hint: You will have to use graphics or a numerical solver to find S_∞ from Eq. (5).

PTO

Case study: On a Pacific island

The population on an idyllic Pacific island is stable at $N = 200$. The birth and death rates are small, and can be ignored. The island's post office employs five people,[39] who are all clustered around when a Christmas hamper is opened for customs inspection. Unfortunately the white powder in the hamper is not artificial snow. Within a day, during which the workers mix as usual with other persons on the island, all five workers are infected with a severe and mysterious virus.

You are the only person on the island with some knowledge of epidemiology, gained from Maths in Year 12. The island's Chief Medical Officer needs to know how many people might need to be treated for the virus and whether to call for emergency hospital facilities. The island's hospital can cope with at most 50 patients at any one time. The CMO asks for your help in predicting the course of the epidemic.

Based on the post-office experience, you assume that the incubation period is less than a day, and can be ignored. The mean infectious period, $1/\alpha$ in the SIR model, needs to be estimated, as does β.

1. Based on other viruses, you take the mean infectious period, $1/\alpha$, to be 5 days and the transmission parameter $\beta = 0.0025$.

   What is the approximate duration of the epidemic $(I < 1)$ with these parameters? Vary *Tmax* to find this accurately.

   Could the hospital cope with the number of patients predicted by this model?

2. After a week you have some actual data. You find that the mean infectious period is actually 4 days, not 5, and that $I(8) = 30$ and $S(8) = 140$.

   Change the value of α, and experiment with different values of β to match this $I(8)$ value.

   What is R_0 for this model?

   What then are your predictions (theoretically and/or numerically based) for

   (a) when the epidemic will peak?

   (b) the duration of the epidemic?

   (c) the number of people who are untouched by the disease at the end?

   Will the hospital be able to cope?

Solutions in Section 14.4.4.

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[39]The island's economy is based around issuing stamps, banking and processing asylum seekers.

14.3.4   Appendix 1: SIR/SIRCE program

Using the programs

1. Run the program. TIME PLOT (below left) plots S, I and R as functions of time. PHASE PLOT plots I as a function of S. INFORMATION gives the SIR equations.

```
 SIR MODEL
1:TIME PLOT
2:PHASE PLOT
3:INFORMATION
4:QUIT
```

```
           TIME PLOT

POPN N: 1000
Tmax: 50
ALPHA: .2
BETA: .0008
S(0): 999
STEP H: 1
```

2. Input the appropriate parameters on the prompts (above right), pressing ⌷enter⌷ after each. Decreasing the step length H will in general improve the accuracy but the calculations will take longer. Starting with a step length of 1 day (H = 1) over 50 days is usually about right. The resulting plot is shown below left.
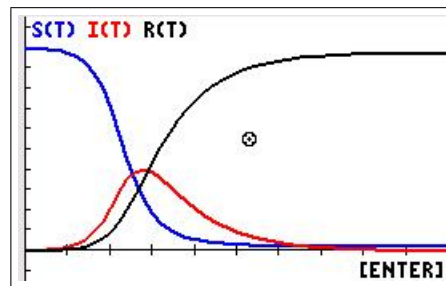

```
FINAL VALUES

T = 50.0
S = 20.8
I = 1.0
R = 978.3
```

3. Press ⌷enter⌷ when the graphs have been plotted to see the final calculated values for S, I and R (above right); press ⌷enter⌷ again to display the NEXT menu (below).

```
 NEXT...
1:NEW H / GRAPH
2:NEW ALPHA,BETA
3:NEW N,S(0)
4:CLEAR GRAPHS
5:CHANGE WINDOW
6:TIME/PHASE
7:QUIT
```

Here you can:

 – change the H value;

 – change the values of α and β;

 – change the values of N and $S(0)$;

 – clear the graphs for another plot; if you don't, the new plot will overlay the current one;

 – change the ⌷window⌷ within the program

 – change from a TIME PLOT to a PHASE PLOT, or vice versa, with the same parameters;

 – QUIT.

14.3.5 Discrete SIR model

14.3.6 The model

If we use a discrete-time model, rather than a continuous-time model (the differential equations of Section 14.3.3), to describe the spread of such a disease, we obtain a SIR model with the difference equations[40]

$$S_{n+1} \;=\; S_n \;-\; \beta S_n I_n \tag{6}$$

$$I_{n+1} \;=\; I_n \;+\; \beta S_n I_n \;-\; \alpha I_n \tag{7}$$

$$R_{n+1} \;=\; R_n \;+\; \alpha I_n, \tag{8}$$

where S_n, I_n and R_n are, respectively, the numbers of susceptible persons, infected persons and recovered persons after n time intervals, and α and β are constants. The time interval here is 1 day.

Case study: On a Pacific island

1. Use the same data that you used in Question 1 of the corresponding continuous model (page 178). Draw graphs (time plots) of S_n and I_n vs n (time in days). Details of how to do this are in Appendix 2 on the next page.

    (a) What is the approximate duration of the epidemic ($I < 1$)? How many persons don't catch the disease?

    (b) When will the peak of infection occur according to the discrete model? Will the hospital cope?

    (c) Show, from the equation for I above, that the theoretical maximum value[41] of I in the discrete model occurs when $S = \alpha/\beta$.

    Compare this value with the value you obtained from your graph. Use $\boxed{\text{trace}}$ to do this. *Why might not these values be exactly the same?*

2. Use the revised value, $\alpha = 0.25$, as in Question 2 of the continuous model (page 178). Again, experiment with different β to find the value that gives $I_8 = 30$.

    Hint: For each β value, use either a graph and $\boxed{\text{trace}}$ or $\boxed{\text{table}}$ [42] (easier) to find I_8 until you obtain the actual value (or as close as possible).

    When you have found the right β value, use the model to predict how many days it will take before the disease dies out and when the peak of infection will occur. *Will the hospital be able to cope according to this model? How many people will avoid catching the disease?*

3. Compare the results from the discrete model with those from the continuous model.

Solutions in Section 14.4.5.

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[40]See Section 14.2 for other examples.

[41]At a maximum or minimum, there is no change in I, so that $I_{n+1} = I_n$. This corresponds to the condition $dI/dt = 0$ in the continuous case.

[42]see the end of Appendix 2

14.3.7   Appendix 2: Sequence graphing

The TI-84/CE calculator has three built-in sequences u, v and w (on the $\boxed{7}$, $\boxed{8}$ and $\boxed{9}$ keys). To access them, press $\boxed{\text{mode}}$; select SEQ (fifth line on the CE, fourth on the 84) with the cursor and press $\boxed{\text{enter}}$. Select 1 decimal place as well (third/second line).

Now press $\boxed{\text{y=}}$ and you will see where to define the sequence functions.

On a **TI-84 Plus**, the sequences $u(n)$ and $v(n)$ are defined in terms of $u(n-1)$, $v(n-1)$, etc. The equations for S and I written in this form are

$$
\begin{aligned}
S_n &= S_{n-1} - \beta S_{n-1} I_{n-1} = S_{n-1}\big(1 - \beta I_{n-1}\big) \\
I_n &= I_{n-1} + \beta S_{n-1} I_{n-1} - \alpha I_{n-1} = I_{n-1}\big(1 + \beta S_{n-1} - \alpha\big) \\
R_n &= R_{n-1} + \alpha I_{n-1}
\end{aligned}
$$

With $S_n \to u(n)$, $I_n \to v(n)$, $R_n \to w(n)$, $\alpha \to A$ and $\beta \to B$, the equations in calculator variables are

$$
\begin{aligned}
u(n) &= u(n-1)\big(1 - Bv(n-1)\big) \\
v(n) &= v(n-1)\big(1 + Bu(n-1) - A\big) \\
w(n) &= w(n-1) + Av(n-1).
\end{aligned}
$$

Enter these equations into the calculator as shown below. Set nMin=0, that is we start with S_0, I_0 and R_0. Then S_1 is the S value after 1 day, etc. The initial conditions are contained in u(nMin, v(nMin) and w(nMin) (=0); these must be entered using curly brackets.

In SEQ mode, the $\boxed{\text{X,T,}\theta\text{,n}}$ key now gives n, the independent variable for the sequence functions. You can't use the letter N.
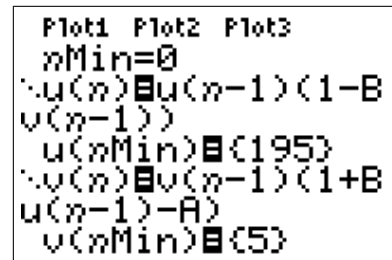
On a **TI-84CE**, the sequences can be input in the original form, Eqs. (6) and (7) (note the top line on the screen below). Set nMin=0, that is we start with S_0 and I_0. Then S_1 is the S value after 1 day, etc. The initial conditions are contained in u(0), v(0) and w(0) (=0).

$$
\begin{aligned}
u(n+1) &= u(n)\big(1 - Bv(n)\big) \\
v(n+1) &= v(n)\big(1 + Bu(n) - A\big) \\
v(n+1) &= w(n) + Av(n)).
\end{aligned}
$$
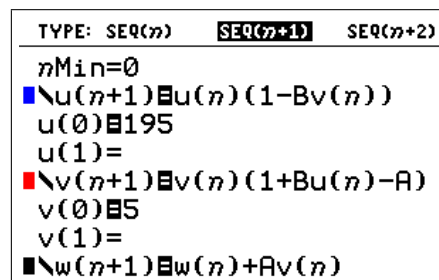
Time plots

Press $\boxed{\text{2nd}}$ $\boxed{\text{format}}$ and select *Time* as shown below.[43] The X axis is n (time in days) and the Y axis S and I.

```
Time Web uv vw uw
RectGC PolarGC
CoordOn CoordOff
GridOff GridDot GridLine
GridColor: MEDGRAY
Axes:  BLACK
LabelOff LabelOn
ExprOn ExprOff
BorderColor:    1
Background:   Off
```

The final step before plotting is to choose a $\boxed{\text{window}}$. As well as the usual window settings for the X and Y axes, we have to specify nMax, the maximum n value we want.

Start by plotting 30 points, corresponding to running the system through 30 days. Note that we must also set Xmax to 30.

As $N = 200$ is the maximum value for both S and I, setting Ymax to 200 seems like a good starting point. Yscl $= 50$.

```
WINDOW
 nMin=0
 nMax=30
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=30
 Xscl=5
 Ymin=0
↓Ymax=200
```

Store the values of α and β in memories A and B respectively. Press $\boxed{\text{graph}}$ to plot graphs of S and I versus n.

Adjust the $\boxed{\text{window}}$, if necessary, until the graphs more or less fill the screen.

Use $\boxed{\text{trace}}$ to explore the values. You can go directly to the point with a particular n value by just typing in the n value and pressing $\boxed{\text{enter}}$.

Adjust nMax and Xmax (both corresponding to Tmax) in $\boxed{\text{window}}$ so that the disease runs its full course ($I < 1$).

You can also see the values of S and I in a table by pressing $\boxed{\text{table}}$ ($\boxed{\text{2nd}}$ $\boxed{\text{graph}}$).

If the n values in the table don't start at 1 and/or don't increment in steps of 1, fix this in $\boxed{\text{tblset}}$ ($\boxed{\text{2nd}}$ $\boxed{\text{window}}$).

---

[43]uv would give a phase plot

14.3.8   Exercise: Age distribution of trees in a forest

This is a similar problem to the SIR model, and can be solved in a similar manner using difference equations and the Sequence grapher. However, the equations here are linear, so that a matrix method can also be used. The matrix version of this problem can be found in *Population Modelling 3: Matrix Models* in Volume 3.

The population of trees in a forest is split into four age groups: b_n is the number of baby trees $(0-15$ years old) at time-point n; y_n the number of young trees $(16-30$ years); m_n the number of middle-aged trees $(31-45$ years old); and o_n the number of old trees (more than 45 years old).

The time step for our difference equations is 15 years.

In order to simplify the model we make the following assumptions:

  A. a certain percentage of trees in each age group dies in each time interval;

  B. surviving trees age into the next age group each time step. Old trees remain old trees (or die);

  C. dead trees are replaced by an equal number of baby trees.

Define α, β, γ, δ as the fraction of dead trees in the respective age groups in each time interval. Then, the difference-equation model is

$$
\begin{aligned}
b_{n+1} &= \alpha b_n + \beta y_n + \gamma m_n + \delta o_n &&\text{(Assumption C)} &&(9)\\
y_{n+1} &= (1-\alpha)b_n &&\text{(Assumptions A, B)} &&(10)\\
m_{n+1} &= (1-\beta)y_n &&\text{(Assumptions A, B)} &&(11)\\
o_{n+1} &= (1-\gamma)m_n + (1-\delta)o_n &&\text{(Assumptions A, B).} &&(12)
\end{aligned}
$$

  1. If the population of trees in time interval n is $N = b_n + y_n + m_n + o_n$, show that the population stays the same size after one more time step, and so by induction the population of trees is a constant N.

     This means that we only need solve Eqs. $(1)-(3)$.

  2. Three initial conditions are needed in order to fully solve these difference equations. Assume all baby trees initially.

     Take $\alpha = 0.2$, $\beta = 0.5$, $\gamma = 0.3$, $\delta = 0.2$ and $N = 1000$.

     Set up Eqs. $(1)-(3)$ with the given parameter values and initial conditions on your calculator. See Section 14.3.7 if you don't know how to do this.

     Run the difference-equation model through 10 cycles (150 years) and plot the results. Do the individual populations appear to be stabilising?

Solutions in Section 14.4.6.

14.4   Solutions

14.4.1   Logistic problem

1. Write down the formula for $P(t)$ and graph it for $0<t<5$.

We have $P_0=10$ and $K=100$, so that $E=P_0/(P_0-K)=-\frac{1}{9}$. After a little algebra,

$$P(t) = \frac{100e^{2.5t}}{9+e^{2.5t}}. \tag{13}$$
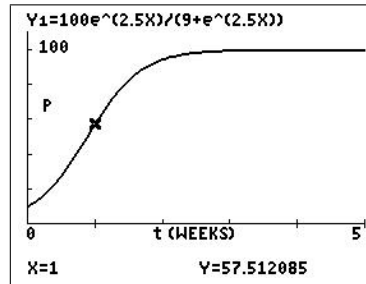
2. How many fish are there after 1 week?

Read directly from your graph using $\boxed{\text{trace}}$ (as in the figure above).

Algebraically, the number of fish after 1 week is, from Eq. (13),

$$P(1) = \frac{100e^{2.5}}{9 + e^{2.5}} = 57.5 \quad \text{to 3 digits}$$

There are 57 or 58 fish in the pond after 1 week.

3. How long does it take the fish population to reach 80?

You can do this from your graph of $P(t)$ by graphing $y=80$, and finding its intersection ($\boxed{\text{calc}}$) with $P(t)$.
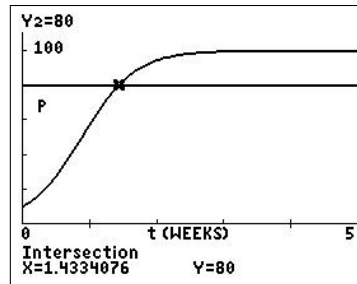
Algebraically, we need to solve $P(t)=80$. Therefore, from Eq. (13),

$$\frac{100e^{2.5t}}{9 + e^{2.5t}} = 80.$$

$$\therefore \quad \frac{e^{2.5t}}{9 + e^{2.5t}} = 0.8.$$

$$\therefore \quad e^{2.5t} = 0.8(9 + e^{2.5t}).$$

$$\therefore \quad 0.2e^{2.5t} = 7.2.$$

$$\therefore \quad e^{2.5t} = 36.$$

$$\therefore \quad t = \frac{\ln(36)}{2.5} = 1.43 \quad \text{to 3 digits.}$$

It takes about 1.43 weeks or about 10 days for the fish population to reach 80.

4. How many fish are there after a long time?

You can do this from your graph of $P(t)$ by tracing along it and remembering what K means.


```
Y1=100e^(2.5X)/(9+e^(2.5X))
  100

  P


  0              t (WEEKS)         5
X=4.9242424        Y=99.995947
```
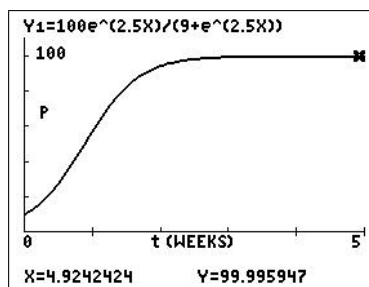
Algebraically, multiplying numerator and denominator of Eq. (13) by $e^{-2.5t}$,

$$P(t) = \frac{100e^{2.5t}}{9 + e^{2.5t}} = \frac{100}{9e^{-2.5t} + 1}.$$

Now $e^{-2.5t}$ goes to 0 as $t \to \infty$, so $P(t) \to 100$. There are 100 fish in the pond after a long time. This is the carrying capacity or maximum sustainable population of the pond according to the logistic model, Eq. (13).

14.4.2   Discrete logistic model

Exercise: Bacteria

For the logistic difference equation $P_{n+1} = AP_n(1 - P_n)$:

with $A = 2$ and starting population $P_0 = 0.1$, the population stabilises at 0.5;


```
1        u(n+1)=2u(n)(1-u(n))


 u(n+1)



 0                n            10
n=7
X=7                    Y=0.5
```
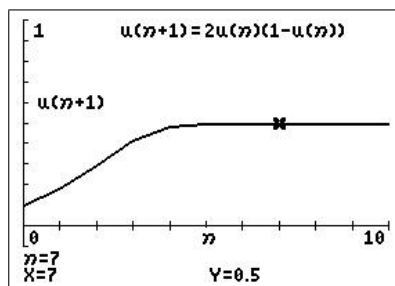
with $A = 3.2$ and $P_0 = 0.1$, eventually the population alternates between 0.513 and 0.799;


```
1



          u(n+1)=3.2u(n)(1-u(n))

0                n              30
n=19
X=19                  Y=0.79943
```

185
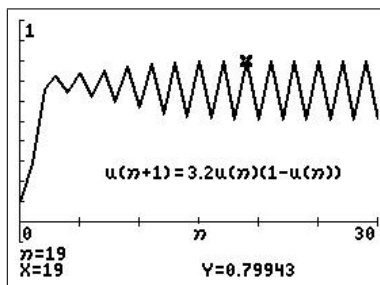
with $A = 3.8$ and $P_0 = 0.1$, the population varies wildly between 0.1 and 1, with no hope of prediction. This one is weird! You've discovered mathematical chaos.
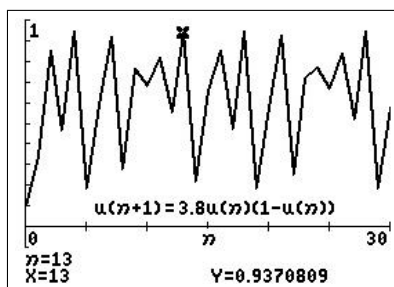
What happens with other values of A and P_0?

Exercise: Kangaroo management

The calculation of the population P_n can be done manually on a calculator (following the calculator hint in the question) or using the TI-84 sequence grapher. The LOGISTIC/LGSTCE program (Section 14.2.3) sets up the sequence grapher for the problem here.

We have the logistic difference equation for the kangaroo population[44]

$$P_{n+1} = 1.8P_n - 0.8(P_n)^2,$$

with $P_0 = 0.8$ corresponding to the (end of) year 2005.

Using this and incorporating the rootoxis outbreak in 2006 by subtracting 0.4 (4000 kangaroos) from the 2006 population, we have the following number of kangaroos in subsequent years.

Year	n	P_n	Number of kangaroos
2005	0	0.8	8000
2006	1	$0.928 - 0.4 = 0.528$	5280
2007	2	0.7274	7274
2008	3	0.8860	8860
2009	4	0.9668	9668

The number of kangaroos has recovered to 9668 by the end of the year 2009. If we include the migration of 2000 kangaroos at the end of 2008, we have the following numbers.

Year	n	P_n	Number of kangaroos
2008	3	$0.8860 + 0.2 = 1.0860$	$10,860$
2009	4	1.0113	$10,113$
2010	5	1.0022	$10,022$
2011	6	1.0004	$10,004$
2012	7	1.0001	$10,001$
2013	8	1.0000	$10,000$

The population in the reserve at the end of 2009 would be $10,113$. In subsequent years, the population would decline to the carrying capacity or maximum sustainable population of $10,000$, the population after a long time.

---

[44]The sequence grapher on the TI-84Plus writes P_n in terms of P_{n-1}, so it is necessary to rewrite the difference equation as $P_n = 1.8P_{n-1} - 0.8(P_{n-1})^2$ if you use this method. The CE does both.

Effect of culling

1. If H kangaroo units are killed each year, this number is subtracted from the value for P_{n+1} that we calculated above, giving the difference equation

$$P_{n+1} = 1.8P_n - 0.8(P_n)^2 - H.$$

2. With $H=0.072$ and an initial population of 0.8 units, the long-term population will be 0.9 units or 9000 kangaroos.
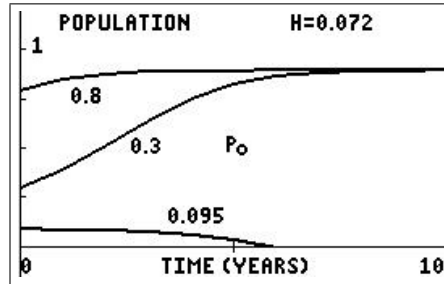
We find that[45] if the initial population is greater than 0.1 kangaroo units, the population will tend toward a stable value of 0.9. If the initial population is less than 0.1 kangaroo units, the population will tend to 0.

3. With $H=0.24$, the population will die out, no matter what the initial population.
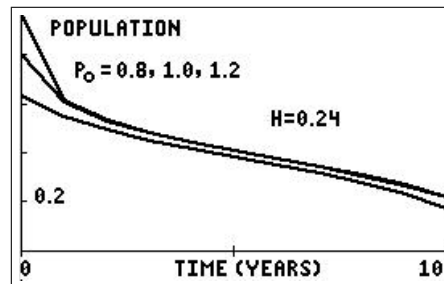
4. With $H=0.2$, the long-term population will be 0.5 units or 5000 kangaroos, the maximum sustainable population with this level of hunting, provided that the initial population is greater than 5000. If the initial population is less than 5000, the population will die out.
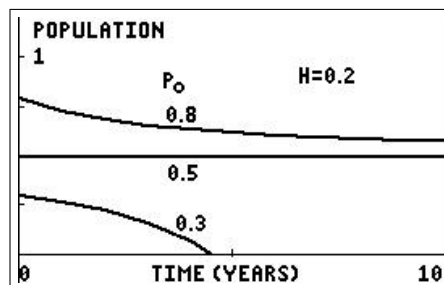
If this level of hunting were chosen, any natural disaster that killed more than a few kangaroos after the population had levelled off at 5000 would bring the population to below 5000, and it would therefore die out. There is no margin for error with this level of hunting. In practice, a value for H smaller than 2000 would be chosen for the number of kangaroos killed annually, thereby leaving a margin to allow for natural disasters.

---

[45]Theory helps a lot here, but you can reach the same conclusions by experimenting with numbers on your calculator. Using the LOGISTIC/LGSTCE program may help with this.

14.4.3   SIR model

1. The final values are as given in the question.
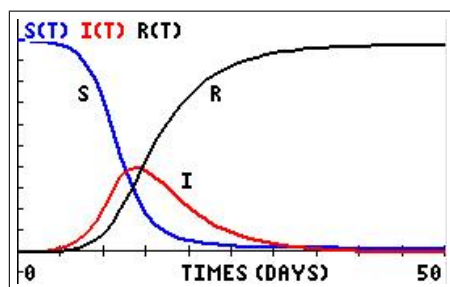
From the graphs, I increases to a maximum, then decreases gradually to 0.

S decreases, slowly at first, then more rapidly, eventually levelling out to some small value (about 20, given that $S(50)=20.8$).

2. Using the SIR/SIRCE program with the parameters as in Question 1 but with a step length H of 0.1 gives the following graph and final values.
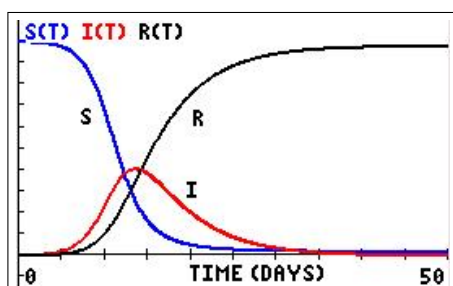
The graphs for S and I, and the values for $S(50)$ and $I(50)$ are the same as in Question 1 when rounded to integer values.

Running SIR/SIRCE with step lengths H of 1, 0.1 and 0.001 days gives the following values after 50 days. We expect the accuracy of the numerical method to increase as we decrease H.

H	$S(50)$	$I(50)$	$R(50)$
1	20.8	1.0	978.3
0.1	19.9	0.9	979.2
0.01	19.9	0.9	979.2

The values change slightly as we decrease the step length, but as we are going to round to integer values, the changes are not significant.

We have, accurate to the nearest integer (0 decimal places):

$S(50) = 20$

$I(50) = 1$

$R(50) = 979$

3. The following table gives the values of S, I and R after 75 days to check if there is any significant change after 50 days. The accuracy of these values is also checked by running the program with steps of 1 day and 0.1 day.

H	$S(75)$	$I(75)$	$R(75)$
1	20.7	0.0	979.3
0.1	19.8	0.0	980.2

The overall result remains the same: 980 people have caught the disease (the size of the epidemic); 20 people do not catch the disease.

After 75 days, the one person still infective after 50 days has recovered.

4. We first keep $\alpha = 0.2$ and vary β. The table below shows the values of $S(50)$, $I(50)$ and $R(50)$ for each value of β. Values rounded to nearest integer.

$\alpha = 0.2$; $S(0) = 999$; $I(0) = 1$; $R(0) = 0$; $H = 1$.
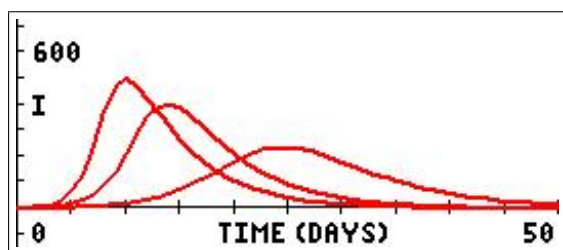
β	$S(50)$	$I(50)$	$R(50)$
0.0005	113	14	873
0.0008	21	1	978
0.0011	5	0	995

The left-hand curve in the figure above has $\beta = 0.0011$, the middle curve $\beta = 0.0008$ and the right-hand curve $\beta = 0.0005$.

From the table and graphs, we see that, the larger the β value, the larger the maximum value of I, the earlier it is reached and the shorter the duration of the epidemic.

The larger the β, the faster the spread of the disease, so we would anticipate the above findings.

5. Now we keep $\beta = 0.0008$ and vary α. The other parameters are as in Question 4.
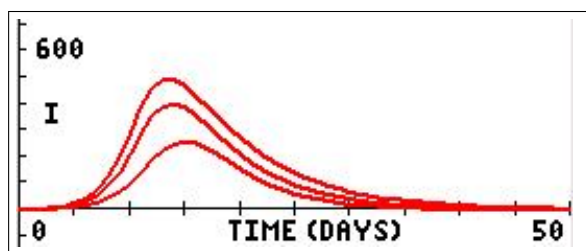
α	$S(50)$	$I(50)$	$R(50)$
0.15	5	4	991
0.2	21	1	978
0.3	89	0	911

The top curve in the figure above has $\alpha = 0.15$, the middle curve $\alpha = 0.2$ and the bottom curve $\alpha = 0.3$.

From the table and graphs, we see that, the larger the α value, the smaller the maximum value of I. The time to maximum I and the duration of the epidemic are both slightly longer with larger α, but the differences are small compared to those seen when changing β in Question 4.

The larger the α, the faster people recover from the disease, so we would expect there to be fewer with the disease at any one time, compared with cases in which α is smaller.

6. We are given that

$$N = S + I + R.$$

$$\therefore \frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt}$$

$$= -\beta SI + \beta SI - \alpha I + \alpha I \qquad \text{from Eqs. (1), (2) and (3)}$$

$$= 0.$$

Therefore, N is a constant. If we know two of the variables, say S and I, we can calculate the third by $R = N - S - I$. Therefore, we only need to work with the DEs involving S and I.

7. Equation (2) is

$$\frac{dI}{dt} = \beta SI - \alpha I$$

$$= I\big(\beta S - \alpha\big)$$

$$= 0 \quad \text{when } I \text{ is a maximum.}$$

As $I \neq 0$ at the maximum, we must then have $S = \alpha/\beta$.

The total number of cases up to this time is then $S(0) - S = S(0) - \alpha/\beta$.

In Questions 1 and 2, $\alpha = 0.2$ and $\beta = 0.0008$, giving the theoretical value of S at maximum I of 250, and the total number of cases by then as $999 - 250 = 749$.

Reading from the graph in Question 1 or 2 (by moving the cursor around), we see that the maximum value of I occurs after about 14 days, with the corresponding S value close to 250, in agreement with the theoretical value.

8. We have

$$\frac{dI}{dt} = \beta SI - \alpha I \qquad \text{for all time } t$$

$$= I\big(\beta S - \alpha\big) \qquad \text{for all time } t$$

$$= I(0)\big(\beta S(0) - \alpha\big) \qquad \text{at } t = 0.$$

If $\beta S(0) - \alpha < 0$ or, equivalently, $S(0) < \alpha/\beta$, then $\dfrac{dI}{dt} < 0$ at $t = 0$.

Therefore I decreases with time from its intial value; there is no epidemic.

Conversely, if $S(0) > \alpha/\beta$, then $\dfrac{dI}{dt} > 0$ at $t = 0$, so that I increases from its initial value; there is an epidemic.

The condition $S(0) < (>) \, \alpha/\beta$ is equivalent to the basic reproductive number $R_0 < (>) \, 1$.

The threshold for an epidemic is therefore $R_0 = 1$, with an epidemic if $R_0 > 1$.

9. Running SIR/SIRCE with $\alpha = 0.2$, $\beta = 0.0008$, $N = 1000$ for 50 days with a step $H = 0.5$ gives the following phase portrait $I(S)$.

Bottom curve: $S(0) = 999$, $I(0) = 1$; middle curve: $S(0) = 900$, $I(0) = 100$; top curve: $S(0) = 800$, $I(0) = 200$. The respective R_0 values are 4.0, 3.6 and 3.2.
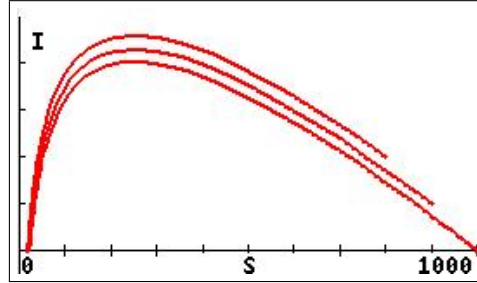
window $[0, 1000, 100] \times [0, 550, 100]$[46]

The graph with a step $H = 0.05$ is very little different, so the curves with $H = 0.5$ are reasonably accurate.

The first plot starts at $(S, I) = (999, 1)$ and initially moves up (I increasing) and to the left (S decreasing). Eventually I peaks at a theoretical value of $S = \alpha/\beta = 250$, and then decreases to 0 at $t \approx 50$. S always decreases. The final point on the graph ($t = 50$) is $(S, I) = (20, 0)$.

The three epidemic graphs differ in initial values, but show very similar behaviour, being roughly parallel until near the end of the epidemic. They appear to have almost the same final S value.

PTO

---

[46]changed from the initial values using the option in the program

10. We have from Eqs. (1) and (2),

$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI - \alpha I.$$

$$\therefore \frac{dI}{dS} = \frac{dI/dt}{dS/dt}$$

$$= \frac{\beta SI - \alpha I}{-\beta SI}$$

$$= -1 + \frac{\alpha}{\beta}\frac{1}{S}.$$

Integrating this with respect to S gives

$$I(S) = -S + \frac{\alpha}{\beta}\ln(S) + D,$$

where D is an arbitrary constant.

If we use t as the independent variable, we can write this as

$$I(t) = -S(t) + \frac{\alpha}{\beta}\ln\big(S(t)\big) + D. \tag{14}$$

The plots here are I versus S. From Question 9, a sensible �window is $[0, 1000, 100] \times [-50, 550, 100]$

Some graphs of $I(S)$ with $\alpha = 0.2$, $\beta = 0.0008$ and various D values are graphed below. They resemble the phase plots in Question 9.

The D values are, top to bottom, -600, -700, -800 and -900.
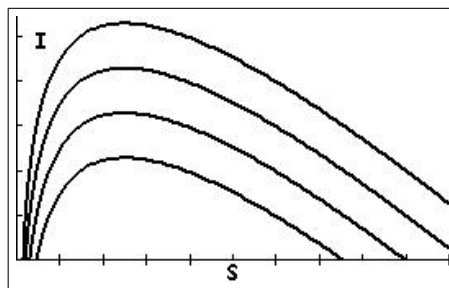
The value of D, for given α, β and N, is set by the initial conditions $S(0)$ and $I(0)$.

Setting $S(0) = 999$, $I(0) = 1$, $\alpha = 0.2$ and $\beta = 0.0008$ in Eq. (14) gives

$$1 = -999 + 250\ln(999) + D.$$

$$\therefore D = 1 + 999 - 250\ln(999)$$

$$= -726.7 \quad \text{rounded to 1 decimal place.}$$

The values for $I(t)$ generated by Eq. (14) are the same values to reasonable accuracy as those generated by SIR/SIRCE.

11. Setting $t=0$ in Eq. (14) and using the notation given in the question results in

$$I_0 = -S_0 + \frac{\alpha}{\beta} \ln (S_0) + D. \tag{15}$$

Similarly, letting $t \to \infty$ in Eq. (14) gives

$$I_\infty = -S_\infty + \frac{\alpha}{\beta} \ln (S_\infty) + D. \tag{16}$$

Subtracting Eq. (15) from Eq. (16) and using a log law gives

$$I_\infty - I_0 = S_0 - S_\infty + \frac{\alpha}{\beta} \ln \left(\frac{S_\infty}{S_0} \right),$$

as required. If we now let $I_0 = I_\infty = 0$, we have

$$0 = S_0 - S_\infty + \frac{\alpha}{\beta} \ln \left(\frac{S_\infty}{S_0} \right).$$

Given $S_0 = S(0)$, we can find S_∞ and hence find $R_\infty = 1000 - S_\infty$, the number of persons who caught the disease. With $S_0 = 999$, $\alpha = 0.2$ and $\beta = 0.0008$, we solve

$$0 = 999 - S_\infty + 250 \ln \left(\frac{S_\infty}{999} \right)$$

numerically for S_∞: graph the right-hand side, replacing S_∞ with X, and find the *relevant* zero.

This gives $S_\infty = 20$, rounded to 0 decimal places.

The size of the epidemic is then $1000 - 20 = 980$, that is 980 persons, the same value we found in Question 3.

PTO

14.4.4 On a Pacific island

1. Use SIR/SIRCE with $N=200$, $S(0)=195$, $I(0)=5$, $\alpha=0.2$, $\beta=0.0025$ and a time of 46 days (after some experimentation). The step length $H=1$ in the following graph, but $H=0.1$ gives the same values when rounded to the nearest integer.
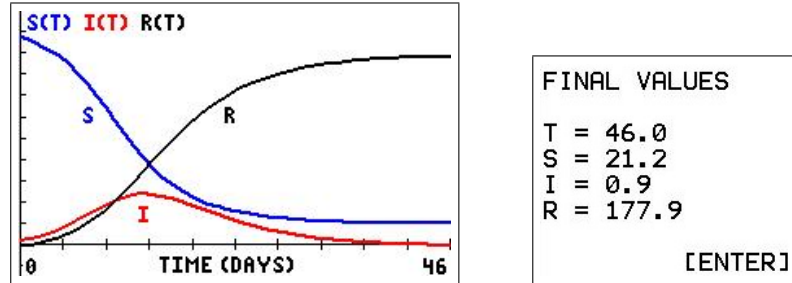
We have $S(46)=21.2$, $I(46)=0.9$ and $R(46)=177.9$.

The duration of the epidemic is about 46 days, the maximum number of infected persons (reading from the graph) is about 49 on about Day 13.

If we vary Xmax (Tmax) to find a more accurate value for $I_{\max}$ (with $H=0.1$), we find that the maximum number of infected persons is 48.7, rounded to 49, on Day 13.

The hospital will therefore just cope.

An algebraic method to find $I_{\max}$

If we evaluate Eq. (14) at $S = S_0$, we obtain

$$I_0 \; = \; -S_0 \; + \; \frac{\alpha}{\beta} \ln\left(S_0\right) \; + \; D. \tag{17}$$

Next, evaluate Eq. (14) at $S = S_{\max}$, the value of S when I is a maximum: $S_{\max} = \alpha/\beta$.

$$I_{\max} \; = \; -S_{\max} \; + \; \frac{\alpha}{\beta} \ln\left(S_{\max}\right) \; + \; D. \tag{18}$$

Subtracting Eq. (17) from Eq. (18) and using a log law gives the following various expressions for $I_{\max}$:

$$
\begin{aligned}
I_{\max} - I_0 \; &= \; S_0 \; - \; S_{\max} \; + \; \frac{\alpha}{\beta} \ln\left(\frac{S_{\max}}{S_0}\right) \\[2mm]
&= \; S_0 \; - \; \frac{\alpha}{\beta} \; + \; \frac{\alpha}{\beta} \ln\left(\frac{\alpha}{\beta S_0}\right) \\[2mm]
&= \; S_0 \; + \; \frac{\alpha}{\beta}\left(\ln\left(\frac{\alpha}{\beta S_0}\right) - 1\right) \\[2mm]
&= \; S_0 \; + \; \frac{S_0}{R_0}\left(\ln\left(\frac{1}{R_0}\right) - 1\right) \\[2mm]
&= \; S_0\left(1 \; - \; \frac{\ln(R_0) + 1}{R_0}\right).
\end{aligned}
$$

For the problem here, $S_0 = 195$, $I_0 = 5$, $\alpha = 0.2$ and $\beta = 0.0025$.

Therefore, $R_0 = \beta S_0/\alpha \approx 2.44$, so that $I_{\max} \approx 48.7$, the same value we found from the SIR/SIRCE calculations.

2. Use SIR/SIRCE with $N = 200$, $S(0) = 195$, $I(0) = 5$ and $\alpha = 0.25$. Try various β values around 0.002 to find which value gives $I(8) = 30$.

The table below shows the results (all with H $= 0.1$).

β	$I(8)$	$S(8)$
0.002	12.9	169.9
0.003	38.2	124.1
0.00276	30.8	137.7
0.00275	30.5	138.2
0.00274	30.2	138.8
0.00273	**29.9**	**139.3**
0.00272	29.6	139.8

The value of β that gives $I(8)$ closest to 30 is $\beta = 0.00273$. Then,

$$R_0 \;=\; \frac{\beta S(0)}{\alpha} \;=\; \frac{0.00273 \times 195}{0.25} \;\approx\; 2.13.$$

An algebraic method to find β

We can use the algebraic theory from the previous question to find a value for β, given that $S(0) = 195$, $I(0) = 5$, $\alpha = 0.25$, $S(8) = 140$ and $I(8) = 30$.

If, instead of evaluating Eq. (18) at $I_{\max}$, we evaluate it at $t = 8$, following the same steps as above we have

$$I(8) \;-\; I(0) \;=\; S(0) \;-\; S(8) \;+\; \frac{\alpha}{\beta}\,\ln\!\left(\frac{S(8)}{S_0}\right).$$

With the above values of the parameters, we then have

$$30 \;-\; 5 \;=\; 195 \;-\; 140 \;+\; \frac{0.25}{\beta}\,\ln\!\left(\frac{140}{195}\right),$$

giving $\beta = 0.00276$, close to the value $\beta = 0.00273$ we found by experimentation.

We used both the given values, $S(8) = 140$ and $I(8) = 30$, in these calculations, whereas we used only $I(8) = 30$ as the criterion in our experimentation above. The difference in the two β values is negligible in our calculations.

With $\alpha = 0.25$, $\beta = 0.00273$ and plotting over 40 days, we obtain the following graph.


```
FINAL VALUES

T = 40.0
S = 31.3
I = 1.0
R = 167.7
                    [ENTER]
```
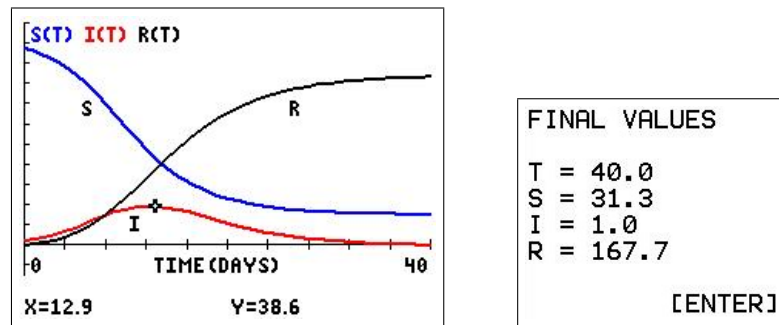
(a) Reading from the graph, the epidemic peaks at about 12 or 13 days, with the maximum number of infected persons about 39.

If we again vary Xmax to find a more accurate value for $I_{\max}$ (with H $= 0.1$), we find that the maximum number of infected persons is 39 on Days 12 and 13. We could also use the algebraic theory of Question 1 to find $I_{\max}$.

(b) The epidemic lasts a little over 40 days $\left(I(40) = 1.0 \right)$.

(c) We find that $S(40) = 31.3$ and $S(50) = 30.9$, so that 31 people don't catch the disease.

The hospital can cope comfortably.

PTO

14.4.5   Discrete approach: On a Pacific island

1. $\alpha = 0.2$, $\beta = 0.0025$. A suitable $\boxed{\text{window}}$ is

```
WINDOW              WINDOW
 nMin=0             ↑nMax=45
 nMax=45             PlotStart=1
 PlotStart=1         PlotStep=1
 PlotStep=1          Xmin=0
 Xmin=0              Xmax=45
 Xmax=45             Xscl=5
 Xscl=5              Ymin=0
 Ymin=0              Ymax=200
↓Ymax=200            Yscl=50
```

with resulting time graph S, I and R vs n (time). The cursor is set on the maximum I value.
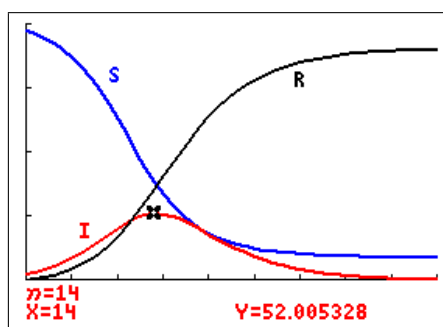
For these values of α and β (using $\boxed{\text{trace}}$ on the calculator graph):

- the maximum I, the peak of infection, of 52 occurs after 14 days. The number of susceptibles then is 76.9;

- the epidemic lasts 44 days, with 18 persons not catching the disease;

- setting $I_{n+1} = I_n = I^*$ for maximum I_n, we have from Eq. (7),

$$I^* = I^* + \beta S_n I^* - \alpha I^*.$$

$$\therefore\ I^*(\beta S_n - \alpha) = 0.$$

$$\therefore\ \beta S_n - \alpha = 0 \qquad \text{as } I^* \neq 0 \text{ at the maximum.}$$

Therefore, at the peak of infection (maximum I_n), we find the same theoretical result as in the continuous case,

$$S_n = \frac{\alpha}{\beta} = \frac{0.2}{0.0025} = 80.$$

From the graph, the value of S at $I_{\max}$, Day 14, is 76.9, close to the theoretical value of 80. The error in the theoretical value is ± 1 day, as S_n is treated as a continuous variable here. Note that $S(13) = 88.2$.

2. A mean infectious period $1/\alpha = 4$ gives $\alpha = 0.25$. We are given $S_8 = 140$, $I_8 = 30$.

The value of $\beta = 0.00281$ gives $S_8 = 143.1$, $I_8 = 30.0$.
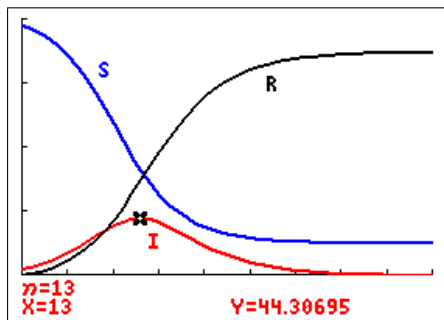
For these values of α and β (again using $\boxed{\text{trace}}$):

- the maximum I of 44 occurs after 13 days, with the corresponding value of S, 81.5, again close to the theoretical value $\alpha/\beta = 89$ (the S value on Day 14 is 93);

- the disease runs for 37 days ($I_{37} < 1$);

- 25 persons (S_{37}) avoid catching the disease.

The hospital therefore just copes.

3. Draw up a table of corresponding values for Questions 1 and 2.

In all cases, $N = 200$, $S(0) = 195$, $I(0) = 5$.

Question 1 $\alpha = 0.2$, $\beta = 0.0025$, $\alpha/\beta = 80$

Quantity	Continuous	Discrete
$I_{\max}/\text{Day}/S$	49/13/69	52/14/77
Duration	46	44
Uninfected	21	18

Question 2 $\alpha = 0.25$

Quantity	Continuous	Discrete
β	0.00273	0.00281
α/β	91.6	81.5
$I_{\max}/\text{Day}/S$	39/12/94	44/13/82
Duration	41	37
Uninfected	31	25

14.4.6   Age distribution of trees in a forest

1. If the population of trees in time interval n is $N = b_n + y_n + m_n + o_n$, show that the population stays the same size after one more time step, and so by induction the population of trees is a constant N.

   Adding the four equations gives

   $$\begin{aligned}
   b_{n+1} + y_{n+1} + m_{n+1} + o_{n+1} &= \alpha b_n + \beta y_n + \gamma m_n + \delta o_n + (1-\alpha)b_n \\
   &\quad + (1-\beta)y_n + (1-\gamma)m_n + (1-\delta)o_n \\
   &= b_n + y_n + m_n + o_n \\
   &= N.
   \end{aligned}$$

   Therefore, the population of trees stays the same size after one more time step, and so by induction the population of trees is a constant N.

   Therefore, $o_n = N - \big(b_n + y_n + m_n\big)$, and we need only solve three equations.

2. Three initial conditions are needed in order to fully solve these difference equations. Assume all baby trees initially.

   Take $\alpha = 0.2$, $\beta = 0.5$, $\gamma = 0.3$, $\delta = 0.2$ and $N = 1000$.

   Set up Eqs. (1)–(3) with the given parameter values and initial conditions on your calculator.

   Run the difference-equation model through 10 cycles (150 years) and plot the results. Do the individual populations appear to be stabilising?

   Substituting for o_n in Eqs. (1)–(3) gives

   $$\begin{aligned}
   b_{n+1} &= \alpha b_n + \beta y_n + \gamma m_n + \delta\big(N - (b_n + y_n + m_n)\big) \\
   &= (\alpha - \delta)b_n + (\beta - \delta)y_n + (\gamma - \delta)m_n + \delta N \\
   y_{n+1} &= (1-\alpha)b_n \\
   m_{n+1} &= (1-\beta)y_n.
   \end{aligned}$$

   With $A = \alpha$, $B = \beta$, $C = \gamma$ and $D = \delta$, and using the sequences u, v and w on the calculator, the three equations become

   $$\begin{aligned}
   u(n+1) &= (A-D)u(n) + (B-D)v(n) + (C-D)w(n) + DN \\
   v(n+1) &= (1-A)u(n) \\
   w(n+1) &= (1-B)v(n).
   \end{aligned}$$

PTO

TI-84CE

The three equations in *Seq* mode are shown in the figure below left. We leave the parameters in as A, B, C, D and N, as this makes it easy to change their values in further exploration of the model.

The initial conditions (all baby trees) are $u(0) = 10000$, $v(0) = 0$ and $w(0) = 0$.

```
Plot1   Plot2   Plot3            WINDOW
TYPE: SEQ(n)  SEQ(n+1)  SEQ(n+2)  ↑nMax=10
■\u(n+1)☰(A-D)u(n)+(B-D)v(        PlotStart=1
n)+(C-D)w(n)+DN                   PlotStep=1
 u(0)☰1000                        Xmin=0
 u(1)=                            Xmax=10
■\v(n+1)☰(1-A)u(n)                Xscl=1
 v(0)☰0                           Ymin=0
 v(1)=                            Ymax=1000
■\w(n+1)☰(1-B)v(n)                Yscl=100
```

The window has n (X) from 0 to 10 cycles and Y from 0 to 1000 trees (above right). Set $A = 0.2$, $B = 0.5$, $C = 0.3$, $D = 0.2$, $N = 1000$ and graph (below left).
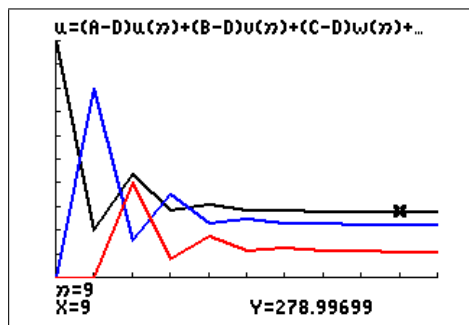
n	u	v	w
5	286.72	250.88	115.2
6	286.78	229.38	125.44
7	281.36	229.43	114.69
8	280.3	225.09	114.71
9	279	224.24	112.54
10	278.53	223.2	112.12
11	278.17	222.82	111.6
12	278.01	222.54	111.41
13	277.9	222.4	111.27
14	277.85	222.32	111.2
15	277.82	222.28	111.16

The three populations (table above right) appear to be stabilising at values around 278 baby trees (black), 222 young trees (blue) and 111 middle-aged trees (red). This gives 389 old trees.

PTO

TI-84

The three equations on a TI-84 have to be written as

$$u(n) = (A-D)u(n-1) + (B-D)v(n-1) + (C-D)w(n-1) + DN$$

$$v(n) = (1-A)u(n-1)$$

$$w(n) = (1-B)v(n-1).$$

The three equations in *Seq* mode are shown in the figure below. We leave the parameters in as A, B, C, D and N, as this makes it easy to change their values in further exploration of the model.

The initial conditions (all baby trees) are $u(0) = 10000$, $v(0) = 0$ and $w(0) = 0$ (note curly brackets).

```
Plot1 Plot2 Plot3        Plot1 Plot2 Plot3
 nMin=0                   u(nMin)⊟{1000}
\u(n)⊟(A-D)u(n-1         \v(n)⊟(1-A)u(n-1
)+(B-D)v(n-1)+(C          )
-D)w(n-1)+DN               v(nMin)⊟{0}
 u(nMin)⊟{1000}          \w(n)⊟(1-B)v(n-1
\v(n)⊟(1-A)u(n-1          )
)                         w(nMin)⊟{0}
```

The WINDOW has n (X) from 0 to 10 cycles and Y from 0 to 1000 trees.

```
WINDOW                   WINDOW
 nMin=0                  ↑PlotStep=1
 nMax=10                  Xmin=0
 PlotStart=1              Xmax=10
 PlotStep=1               Xscl=1
 Xmin=0                   Ymin=0
 Xmax=10                  Ymax=1000
↓Xscl=1                   Yscl=100
```

Set $A = 0.2$, $B = 0.5$, $C = 0.3$, $D = 0.2$, $N = 1000$ and graph (below left).
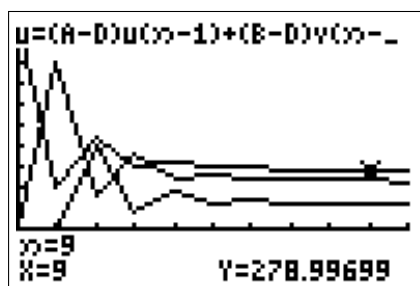
```
u=(A-D)u(n-1)+(B-D)v(n-_        n  | u(n)  | v(n)
                               9   | 279   | 224.24
                               10  | 278.53| 223.2
                               11  | 278.17| 222.82
                               12  | 278.01| 222.54
                               13  | 277.9 | 222.4
 n=9                           14  | 277.85| 222.32
 X=9        Y=278.99699        15  |▓▓▓▓▓▓▓| 222.28
                               u(n)=277.8167442
```

The three populations (table above right) appear to be stabilising at values around 278 baby trees (top graph), 222 young trees (middle graph) and 111 middle-aged trees (bottom graph). This gives 389 old trees.

15   Multivariable Calculus

15.1   Introduction

As with all new topics, students should first learn to carry out calculations by hand until they have mastered the process. The material presented here is to serve two purposes: one is to check hand calculations, the other to provide a method to do calculations which are difficult to do by hand or which are only a small part of a larger calculation in, say, a modelling project. The use of graphics calculators (really palmtop computers), as with computers, allows us to proceed to interesting parts of a project without getting bogged down in lengthy hand calculations. Calculations in multi-variable Calculus can very quickly become complicated, leading to enthusiasm-sapping mistakes.

The derivative command on the TI-84/CE, *nDeriv*, is used to calculate (approximations to) the values of the first and second derivatives of a function $f(x)$ at a given value $x = a$: $f'(a)$ and $f''(a)$.[47] The designers of the TI-84/CE have also made it possible to use *nDeriv* to calculate (approximations to) the values of the first and second partial derivatives of a function $f(x, y, z)$ at a given point (a, b, c):[48] $f_x(a, b, c)$; $f_y(a, b, c)$; $f_z(a, b, c)$; $f_{xx}(a, b, c)$; $f_{yy}(a, b, c)$; $f_{zz}(a, b, c)$; and the six mixed second partial derivatives. Here, we use this facility to calculate the first and second partial derivatives, the gradient and the directional derivative of a scalar function of three variables, and the divergence and curl of a 3D vector function of three variables.

There is also a numerical integrator *fnInt* on the TI-84/CE to integrate functions of one variable. Unfortunately, *fnInt* cannot be nested (like *nDeriv*) to evaluate double integrals, that is to integrate functions of two variables. Double integrals are done here using a combination of *fnInt* and Simpson's Rule.

15.1.1   Calculator note

There are two options on the TI-84/CE that can help with calculations by hand in MVC.

MATHPRINT

In CLASSIC mode, commands are typed on one line, with arguments in brackets. In MATH-PRINT mode, the calculator tries to display commands in mathematical notation, with small boxes in the relevant positions for the inputs. This applies to the powers and roots (blue) commands on the keyboard, to functions and to many of the commands in the $\boxed{\text{math}}$ NUM menu, including the numerical derivative *nDeriv* and numerical integrator *fnInt*.

Set CLASSIC or MATHPRINT in $\boxed{\text{mode}}$. Your choice does not affect the programs here.

Catalog Help

An app on the TI-84Plus,[49] built in to the operating system of a TI-84CE, this tells you what arguments are required for most of the commands on the calculator in CLASSIC mode, allows you to input the arguments and paste the whole command onto the Home screen. Activated by scrolling down to the particular command and pressing $\boxed{+}$.

---

[47]See Chapter 11 for details.

[48]The method can be extended easily to functions of more than three variables.

[49]download from *education.ti.com*

15.1.2   MVCCALC/MVCCALCE program

The program[50] carries out all the operations listed in the following sections on functions of three variables, no matter how complicated the function, taking the hard work out of the calculations. The program is available from *www.XXX*.

Select the appropriate operation from the main menu.

INFO gives information on the types of functions involved and where they are to be stored in the calculator.

```
█MVC 3D OPERATIONS█
█:INFO
2:PARTIAL DERIVATIVES
3:GRAD F
4:DIRECTIONAL DERIVATIVE
5:DIV F
6:CURL F
7:DOUBLE INTEGRATION
8:QUIT
```

The programs can also be used for functions of two variables (or even one), where appropriate, by inputting zeros in the appropriate place(s).

15.2   Partial derivatives

15.2.1   Calculating partial derivatives

The built-in numerical differentiator *nDeriv* ($\boxed{\text{math}}$ $\boxed{8}$) uses the symmetric or central difference quotient to approximate the derivative of a function at a given point.[51] In CLASSIC mode, the command is of the form

$$\text{nDeriv}\big(\textit{function, variable, value}[,\text{H}]\big),$$

where H ($= \Delta X = \Delta Y$) is an optional accuracy parameter, with default value 10^{-5}.

In MATHPRINT mode, the command is displayed as you would write it, with small boxes for input of the indepent-variable name, the function to be differentiated and the value of the indepent variable at which the derivative is to be evaluated. You cannot vary H in this form: the default value is used.

MVCCALC/MVCCALCE uses *nDeriv* to calculate (approximations to) the first partial derivatives of a function of up to three variables $f(X, Y, Z)$ at a given point (A, B, C), and two nested *nDeriv* commands for the second derivatives.[52] More details in Section 15.4.

PTO

---

[50]MVCCALC for the TI-84Plus, MVCCALCE for the TI-84CE

[51]See *Graphing and Calculus Operations*, Chapter 11, for more details.

[52]If f is a function of two variables only, put $C = 0$. You need to input a value for C or the program won't proceed but, with no z in the function, the value can actually be anything you like.

Use: Type the function $f(X, Y, Z)$ into Y₁. Run the program and follow the prompts. Press $\boxed{\text{enter}}$ to generate successive approximations.

In the program, the value of the tolerance H, displayed on the screen, decreases by a factor of 10 each time you press $\boxed{\text{enter}}$,[53] starting with H $= 0.1$.[54]

Press the $\boxed{\text{S}}$ key (no $\boxed{\text{alpha}}$ required) to exit the calculation. This takes you to a menu where you can choose a different derivative, a different point, a different operation or quit the program.

Example: Calculate the first and second partial derivatives of $f(x, y, z) = \dfrac{x^3 \sin(y)}{z+1}$ at the point $(3, 2, 1)$.

Set Y₁ $=$ X³sin(Y)/(Z+1), run the program, select PARTIAL DERIVATIVES, input the point $(3, 2, 1)$ and select FX (f_x). Each time you press $\boxed{\text{enter}}$, the answer is recalculated with H decreased by a factor of 10.

```
 MVC 3D OPERATIONS
1:INFO
2:PARTIAL DERIVATIVES
3:GRAD F
4:DIRECTIONAL DERIVATIVE
5:DIV F
6:CURL F
7:DOUBLE INTEGRATION
8:QUIT
```

```
Y₁=X³sin(Y)/(Z+1)
```

```
 PARTIAL DERIVATIVES OF
 Y₁=F(X,Y,Z) AT (A,B,C)

A=?3
B=?2
C=?1
```

```
 DERIVATIVE
1:FX
2:FY
3:FZ
4:FXX
5:FYY
6:FZZ
7:MIXED SECOND DERIVATIVES
8:QUIT
```

```
FX AT {3,2,1}
              12.28006175
H
0.1



[ENTER]
[S]:STOP
```

```
FX AT {3,2,1}
              12.28006175
H             12.27556073
0.01



[ENTER]
[S]:STOP
```

```
FX AT {3,2,1}
              12.28006175
H             12.27556073
1ε-5          12.27551572
              12.27551527
               12.2755153


[ENTER]
......        [S]:STOP
```

```
 WHERE NEXT?
1:NEW DERIVATIVE
2:NEW POINT
3:NEW OPERATION
4:QUIT
```

Press the $\boxed{\text{S}}$ key to end the calculations and display a menu of options (right-hand figure above).

PTO

---

[53]In general, the smaller H, the more accurate the answer, up to a point. If H is too small, round-off error in the calculation will eventually result in reduced accuracy. This should be obvious in the sequence of answers as H is reduced, usually as a rapid decrease in the number of decimal places in the answers.

[54]On a TI-84, but not on a CE, the calculations stop at H $= 10^{-5}$, usually small enough to guarantee an answer accurate to a reasonable number of significant digits.

The approximations to the derivatives, all with $H = 10^{-4}$, checked with $H = 10^{-5}$, have been rounded here and are accurate to 4 significant digits:

$$f_x(3, 2, 1) \ = \ 12.28$$

$$f_y(3, 2, 1) \ = \ -5.618$$

$$f_z(3, 2, 1) \ = \ -6.138$$

$$f_{xx}(3, 2, 1) \ = \ 8.184$$

$$f_{yy}(3, 2, 1) \ = \ -12.28$$

$$f_{zz}(3, 2, 1) \ = \ 6.138$$

$$f_{xy}(3, 2, 1) \ = \ f_{yx}(3, 2, 1) \ = \ -5.618$$

$$f_{xz}(3, 2, 1) \ = \ f_{zx}(3, 2, 1) \ = \ -6.138$$

$$f_{yz}(3, 2, 1) \ = \ f_{zy}(3, 2, 1) \ = \ 2.809$$

These are all the same to 4 significant digits as the exact values, calculated algebraically.

15.2.2   Gradient

The 3D del, nabla or or gradient vector operator

$$\underset{\sim}{\nabla} \ \equiv \ \left(\frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ \frac{\partial}{\partial z} \right)$$

is the multivariable Calculus equivalent of d/dx. Operating on a scalar function $f(x, y, z)$ with $\underset{\sim}{\nabla}$ gives the gradient or gradient vector, $\operatorname{grad} f$, as

$$\operatorname{grad} f \ = \ \underset{\sim}{\nabla} f \ = \ \left(\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y}, \ \frac{\partial f}{\partial z} \right).$$

MVCCALC/MVCCALCE calculates (approximations to) the first partial derivatives in the gradient of a function of three variables $f(X, Y, Z)$ at a given point (A, B, C) using the *nDeriv* command with $H = 10^{-4}$. The values are rounded to 4 decimal places. You can scroll along the answer if it does not all fit on one screen.

Use: Type the function $f(X, Y, Z)$ into Y1. Run the program, and enter A, B and C.
For a function of two variables $f(X, Y)$, set $C = 0$.

Example: Find $\operatorname{grad} f$ at the point $(1, 2, 3)$ for $f(x, y, z) = x^2 \sin(y) e^z$.

```
          GRAD F AT (A,B,C)

     A=?1
     B=?2
     C=?3

     GRAD F =
      {36.5275  -8.3585 18.2637}

                         [ENTER]
```

```
∎\Y₁=X²sin(Y)e^(Z)
```

15.2.3   Directional derivative

The directional derivative of a function f in the direction of a vector $\boldsymbol{u}$, that is the rate of change of f in the direction of $\boldsymbol{u}$, is given by the dot or scalar product

$$f_{\boldsymbol{u}} \;=\; \underset{\sim}{\boldsymbol{\nabla}} f \cdot \hat{\boldsymbol{u}},$$

where $\hat{\boldsymbol{u}}$ is the unit vector in the direction of $\boldsymbol{u}$.

MVCCALC/MVCCALCE calculates $f_{\boldsymbol{u}}$ for a function $f(x, y, z)$ at a given point (A, B, C); values are rounded to 4 decimal places. $H = 10^{-4}$ in *nDeriv*.

Use: Type the function $f(X, Y, Z)$ into Y₁. Run the program, enter A, B and C, and the components of $\boldsymbol{u}$ in curly brackets, separated by commas; $\boldsymbol{u}$ does not have to be a unit vector. For a function of two variables $f(X, Y)$, set C and the third component of $\boldsymbol{u}$ to 0.

Example: Find $f_{\boldsymbol{u}}$ for $f(x, y, z) = x^2 \sin(y) e^z$ at the point $(1, 2, 3)$, where $\boldsymbol{u} = (-1, 3, -2)$.

```
DIRECTIONAL DERIVATIVE Fu
AT POINT (A,B,C)
IN DIRECTION U
A=?1
B=?2
C=?3
VECTOR U { , , }
?{-1,3,-2}
Fu = -26.2265
                      [ENTER]
```

15.2.4   Divergence

The divergence of a vector function $\boldsymbol{F}$ is given by

$$\operatorname{div} \boldsymbol{F} \;=\; \underset{\sim}{\boldsymbol{\nabla}} \cdot \boldsymbol{F},$$

a scalar, where the dot indicates a dot or scalar product.

In Cartesian coordinates, if $\boldsymbol{F} = (F_1, F_2, F_3)$, then

$$\operatorname{div} \boldsymbol{F} \;=\; \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

MVCCALC/MVCCALCE calculates $\operatorname{div} \boldsymbol{F}$ at a given point (A, B, C); values are rounded to 4 decimal places. $H = 10^{-4}$ in *nDeriv*.

Use: Type the function $F_1(X, Y, Z)$ into Y₁, $F_2(X, Y, Z)$ into Y₂ and $F_3(X, Y, Z)$ into Y₃. Run the program, and enter A, B and C.
For a function of two variables $f(X, Y)$, set C and Y₃ to 0.

PTO

Example: Find div $\boldsymbol{F}$ at the point $(1, 2, 3)$ for $\boldsymbol{F}(x, y, z) = \left(\ln(x^2+1), \cos(y), xye^z\right)$.

```
        DIV F AT (A,B,C)
          F=(Y₁,Y₂,Y₃)

A=?1
B=?2
C=?3

DIV F = 40.2618

                    [ENTER]
```

```
■\Y₁⊟ln(X²+1)
■\Y₂⊟cos(Y)
■\Y₃⊟XYe^(Z)
```

15.2.5 Curl

The curl of a 3D vector function $\boldsymbol{F}(x, y, z)$ is given by

$$\operatorname{curl} \boldsymbol{F} = \underset{\sim}{\boldsymbol{\nabla}} \times \boldsymbol{F},$$

a vector, where $\times$ indicates the cross or vector product.

MVCCALC/MVCCALCE calculates curl $\boldsymbol{F}$ at a given point (A, B, C); values are rounded to 4 decimal places. $\mathrm{H} = 10^{-4}$ in *nDeriv*.

Use: Type the function $F_1(X, Y, Z)$ into Y₁, $F_2(X, Y, Z)$ into Y₂ and $F_3(X, Y, Z)$ into Y₃. Run the program, and enter A, B and C. You can scroll along the answer if it does not all fit on one screen.

Example: Find curl $\boldsymbol{F}$ at the point $(1, 2, 3)$ for $\boldsymbol{F}(x, y, z) = \left(\ln(x^2+1), \cos(y), xye^z\right)$.

```
       CURL F AT (A,B,C)
          F=(Y₁,Y₂,Y₃)
A=?1
B=?2
C=?3

CURL F =
        {20.0855 -40.1711 0}

                    [ENTER]
```

```
■\Y₁⊟ln(X²+1)
■\Y₂⊟cos(Y)
■\Y₃⊟XYe^(Z)
```

PTO

15.3   Double integrals

15.3.1   The program

MVCCALC/MVCCALCE evaluates (*note the order of integration:* Y *first, then* X)

$$\int_A^B \int_{L(X)}^{U(X)} f(X, Y)\, dY\, dX$$

using the calculator integration command *fnInt* to do the Y integral and Simpson's Rule to do the X integral.

Use: Put the function $f(X, Y)$ in Y1. Run the MVCCALC/MVCCALCE program, select DOUBLE INTEGRATION , read the message and press $\boxed{\text{enter}}$. Select CALCULATE and enter the Y integration limits L(X) and U(X) (Lower and Upper — these are functions of X in general) at the prompts, pressing $\boxed{\text{enter}}$ after each. Then enter the X integration limits A (lower) and B (upper). These must be constants.

The Simpson's Rule starts with $N = 30$ sub-intervals and displays this approximation to the exact answer. It then doubles N, and displays the next approximation. The accuracy improves as N increases, but it takes longer to do the calculation.[55]

The value of N for the last calculation is shown at the bottom left of the screen.

The calculator will try to display the answer as a fraction.

Stop the program by pressing $\boxed{\text{on}}\ \boxed{1}$ (Quit) when the answer has reached the desired accuracy — that is, successive answers are the same to the required number of decimal places/significant digits.

Note: You must calculate at least two successive answers to give you an idea of the accuracy. A single answer may be accurate or it may not. You can't tell. The final answer is stored in variable/memory S.

x integral to be done before the y integral

Swap the x and y symbols wherever they occur in the integral (including in the integration limits). *You do not have to reverse the order of integration.*

If you do work through reversing the order of integration, you will, of course, end up with an integral in the correct form for the program but with much more work.

Polar integrals or integrals in terms of variables other than x and y

In polar integrals, decide which integral to do first, r or θ. Replace this variable with y. Replace the other variable with x.

With variables other than x and y, rewrite the integral using x and y as the variables, with y the variable for the first integration.

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---

[55]The maximum value for N is 960, set by the maximum number of entries in a list (999) on the calculator.

15.3.2   The method

TI calculators have a built-in numerical integrator, *fnInt*, which uses Gauss-Kronrod quadrature, a sophisticated method for numerical integration. In CLASSIC mode, the command is of the form (see Chapter 11 for more details)

$$\text{fnInt}(\textit{function, variable, lower limit, upper limit}[, \textit{tolerance}]).$$

tolerance is an optional accuracy parameter, with default value 10^{-5}.

Unfortunately, you cannot nest two *fnInt* commands[56] to generate a double integral. The MVCCALC/MVCCALCE program uses *fnInt*, with the default tolerance, to carry out the Y (first) integration, then Simpson's Rule for the X (second) integration.

To do the Y integral, the calculator sets $Y_2 = \text{fnInt}(Y_1, Y, Y_3, Y_4)$, where the Y integration limits, stored in Y_3 and Y_4, are functions of X in general. The Simpson's Rule provides the value of X for each of the *fnInt* integrals.

Examples

1. Evaluate $\displaystyle\int_0^1 \int_{x^2}^x (x+y)\, dy\, dx.$

   Set $Y_1 = X+Y$.

   Run the MVCCALC/MVCCALCE program, press $\boxed{\text{enter}}$ after reading the screen, select CALCULATE and enter the Y limits X^2 (Lower) and X (Upper). Then enter the X limits A as 0, B as 1 (figures below).

   Answer: 0.15 or 3/20.

   Although the TI-84CE gives an answer of 0.15 (to 10 decimal places) with $N = 240$, it is only when $N = 480$ that it recognises that the answer is exactly 3/20 (it obviously requires agreement with the fraction to at least 11 decimal places).[57]

```
           DOUBLE INTEGRAL

      INTEGRATES F(X,Y) IN Y₁.

      Y INTEGRAL IS DONE FIRST.
      Y LIMITS CAN BE FUNCTIONS
      OF X.
      PRESS [ON] [1] TO STOP.

                        [ENTER]
```

```
■\Y₁■X+Y
```

```
        DOUBLE INTEGRAL
Y LIMITS
L(X): X²
U(X): X
X LIMITS
A=?0
B=?1
```

```
A=?0
B=?1
   N           INTEGRAL
  30        0.1499999177
  60        0.1499999949
 120        0.1499999997
 240                0.15
 480                3/20
```

   Press $\boxed{\text{on}}$ $\boxed{1}$ (Quit) when the answer has reached the desired accuracy.

---

[56]like we did with *nDeriv* commands to calculate second derivatives

[57]The TI-84 gives the answer 3/20 with $N = 240$, indicating that it only requires agreement with the fraction to 10 decimal places.

2. Evaluate $\displaystyle\int_0^1 \int_{y^2}^1 (x-y)\, dx\, dy$.

Rewrite as $\displaystyle\int_0^1 \int_{x^2}^1 (y-x)\, dy\, dx$ swap the x and y symbols.

Set $Y_1 = Y - X$.

Run the program, enter the Y limits X^2 and 1, and the X limits 0 and 1.

Answer: Again 0.15 or 3/20.

Here it again achieves the exact answer 3/20 when $N = 480$.

3. Evaluate $\displaystyle\int_0^1 \int_0^{\pi/2} r^2 \cos(\theta^3)\, d\theta\, dr$.

Rewrite as $\displaystyle\int_0^1 \int_0^{\pi/2} x^2 \cos(y^3)\, dy\, dx$ $\theta \to y,\ r \to x$.

Set $Y_1 = X^2 \cos(Y^3)$.

Run the program, enter the Y limits 0 and $\pi/2$, and the X limits 0 and 1.

Answer: 0.2339, accurate to 4 significant digits.

15.4   Technical note

There are one or two subtleties in calculating partial derivatives, especially second partial derivatives, using *nDeriv*. In CLASSIC mode, the command is of the form

$$\text{nDeriv}(\textit{function, variable, value}[,\ \text{H}]),$$

where H $(= \Delta X = \Delta Y)$ is an optional accuracy parameter, with default value 10^{-5}. Here we have used *nDeriv* to calculate (approximations to) the first partial derivatives of a function of (up to) three variables $f(X, Y, Z)$ at a given point (A, B, C), and two nested *nDeriv* commands for the second derivatives.

The first partial derivatives are straight forward, the only subtlety being that the values A, B and C must also be stored in variables X, Y and Z before a calculation so that the two variables not involved in a differentiation take on the correct values. The value for the variable involved in the differentiation is assigned in the *nDeriv* command. Hence, with the function f in Y_1, with A, B and C stored in variables X, Y and Z, and setting $H = 10^{-4}$, the CLASSIC command for:

f_x is $\text{nDeriv}(Y_1,\ X,\ A,\ H)$; f_y is $\text{nDeriv}(Y_1,\ Y,\ B,\ H)$; f_z is $\text{nDeriv}(Y_1,\ Z,\ C,\ H)$.

The second partial derivatives are more complicated. The CLASSIC command for

f_{xx} is $\text{nDeriv}(\text{nDeriv}(Y_1, X, X, H),\ X,\ A,\ H)$, with the values for the second X in the inner command supplied by the outer command.

Similarly, the command for

f_{yy} is $\text{nDeriv}(\text{nDeriv}(Y_1, Y, Y, H),\ Y,\ B,\ H)$, and for

f_{zz} is $\text{nDeriv}(\text{nDeriv}(Y_1, Z, Z, H),\ Z,\ C,\ H)$.

The mixed second partial derivatives are slightly different because the differentiations are with respect to different variables. The CLASSIC command for

f_{yx} is $\mathrm{nDeriv}\big(\mathrm{nDeriv}(Y_1, Y, B, H), X, A, H\big)$, for

f_{zx} is $\mathrm{nDeriv}\big(\mathrm{nDeriv}(Y_1, Z, C, H), X, A, H\big)$, and for

f_{zy} is $\mathrm{nDeriv}\big(\mathrm{nDeriv}(Y_1, Z, C, H), Y, B, H\big)$.

The three remaining mixed second partial derivatives are given by $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$ and $f_{yz} = f_{zy}$.

You can also nest derivative commands in MATHPRINT mode in a similar manner (but with the default H), as in the figure below, which uses the example on page 204.
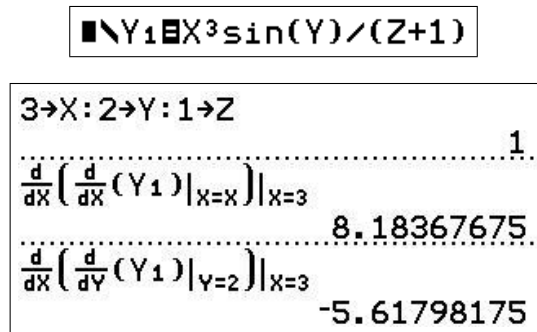
15.5   Exercises with answers

Answers are given to 4 significant digits, with the exact answers in brackets.

1. Find all the first and second derivatives of $f(x, y, z) = x^2 e^y / z$ at the point $(3, 2, 1)$, accurate to 4 significant digits.

   Answers: $f_x = 44.33$ $(6e^2)$; $f_y = 66.50$ $(9e^2)$; $f_z = -66.50$; $f_{xx} = 14.78$ $(2e^2)$; $f_{yy} = 66.50$; $f_{zz} = 133.0$ $(18e^2)$; $f_{yx} = f_{xy} = 44.33$; $f_{zx} = f_{xz} = -44.33$; $f_{zy} = f_{yz} = -66.50$.

2. For $f(x, y, z) = \tan(x) + y^2 \sqrt{z}$, find $\mathrm{grad}\, f$ at the point $(0, 3, 2)$, and the directional derivative there in the direction of $\boldsymbol{u} = (-1, 2, 4)$.

   Answers: $\mathrm{grad}\, f = (1,\ 8.485,\ 3.182)$ $\left(\left(1,\ 6\sqrt{2},\ 2.25\sqrt{2}\right)\right)$; $f_{\boldsymbol{u}} = 6.263$ $\left(\frac{21\sqrt{2}-1}{\sqrt{21}}\right)$.

3. Find $\mathrm{div}\,\boldsymbol{F}$ and $\mathrm{curl}\,\boldsymbol{F}$ for the vector function $\boldsymbol{F} = \big(xyz, 2y^2 \ln(z), 6z^2 \sin(x)\big)$ at the point $\left(\frac{\pi}{2}, 3, 1\right)$.

   Answers: $\mathrm{div}\,\boldsymbol{F} = 15$ (15); $\mathrm{curl}\,\boldsymbol{F} = (-18,\ 4.712,\ -1.571)$ $\left(\left(-18,\ \frac{3\pi}{2},\ -\frac{\pi}{2}\right)\right)$.

4. Find the following integrals.

   (a) $\displaystyle\int_0^2 \int_0^x e^{x^2}\, dy\, dx$.

      Answer: 26.80 $\left(\frac{1}{2}(e^4 - 1)\right)$.

(b) $\displaystyle\int_1^4 \int_{\sqrt{y}}^{y} x^2 y^3 \, dx \, dy.$

Answer: 656.1 $\left(\frac{1}{3} \left(\frac{4^7 - 1}{7} - \frac{2(4^{5.5} - 1)}{11} \right) \right).$

(c) $\displaystyle\int_0^{\pi/6} \int_0^{2/\cos(\theta)} r \, dr \, d\theta.$

Answer: 1.155 $\left(2 \tan \left(\frac{\pi}{6} \right) \right).$

5. Do all the above questions algebraically to check your answers.

16   Calculator Programs

16.1   Why use programs?

While the first step in learning any new mathematical technique is doing it by hand until you are competent and understand the proceedure, eventually doing all calculations by hand becomes tedious, particularly if the calculation is lengthy or complex (in the general sense). This is a barrier to the next (and more interesting) step, using the technique to explore some mathematical entity or to model some process. This usually entails repeated calculations with different values of the parameters of the problem; with the best will in the world, no-one is going to do much of this by hand. Much more interesting is to generate results, numbers or graphs, that one can interpret in the context of the problem.

This is where a program comes in: it does the same steps you would do by hand, but much much faster and probably more accurately. The program can also generate the form of output best suited to the task. You can enter the world of *what if?* and *what does this mean?*. This is the intention of the programs here, not as black boxes but as technology to do the calculations you could do by hand.

A fringe benefit of using the programs is, I hope, an encouragement to write your own programs (coding) for the same reason. In my (long) experience, writing programs for the TI-84/CE is one of the best ways to learn coding. See *Programming* in Volume 3 for more on this and an introduction to programming.

16.2   Copying programs

16.2.1   From the website

You will need either *TI Connect* or *TI Connect CE* for this. If your TI-84 has a mini-USB port,[58] use *TI Connect CE*, otherwise use *TI Connect*. Both are available for free at *education.ti.com*. Click on *Downloads* and follow through.

To copy a program from the website, open up/unzip the program folder on the website and drag the program you want (see Section 16.4 *Program Protection* below) onto your desktop or into another folder.

Connect your calculator to your computer with the appropriate cord and start *TI Connect* or *TI Connect CE*.

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[58]all CE and 84Plus calculators except for some early model 84Plus

TI Connect CE

TI Connect CE should recognise your calculator. Click on the central icon of the three icons in the left-hand column; this is for transfers between computer and calculator.[59]
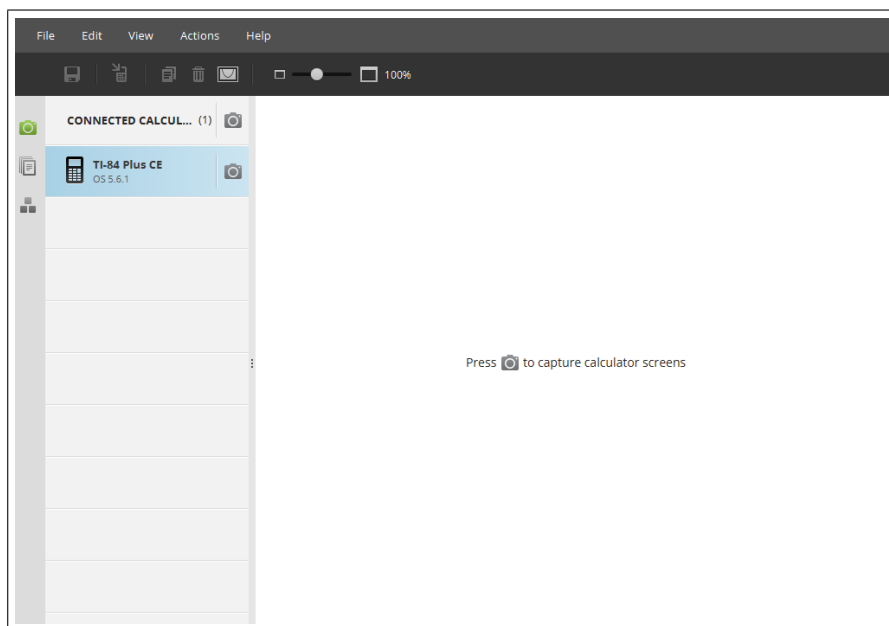
As in the screenshot below,[60] the contents of your calculator should be shown.
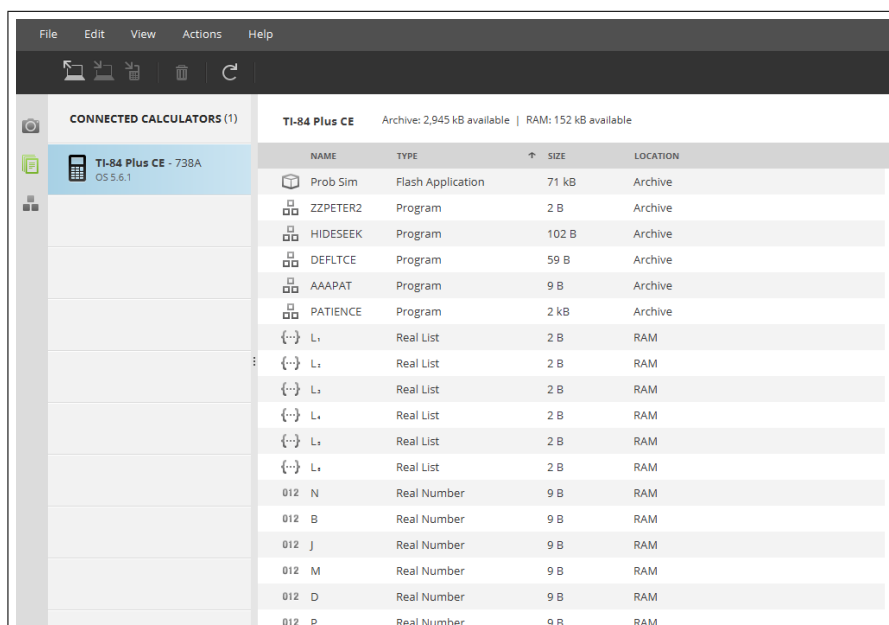
On the left of the second row of icons at the top of the screen is an icon for copying programs from your computer (computer with arrow facing away); click on this.

---

[59]The top icon is for downloading calculator screen shots (the option shown), the bottom icon for editing programs.

[60]taken from a PC running Windows 7

Select the desktop or folder where you have copied the program, then click on the program (screenshot below). Click on *Open* (bottom right of the screen) and follow the prompts to complete the transfer.
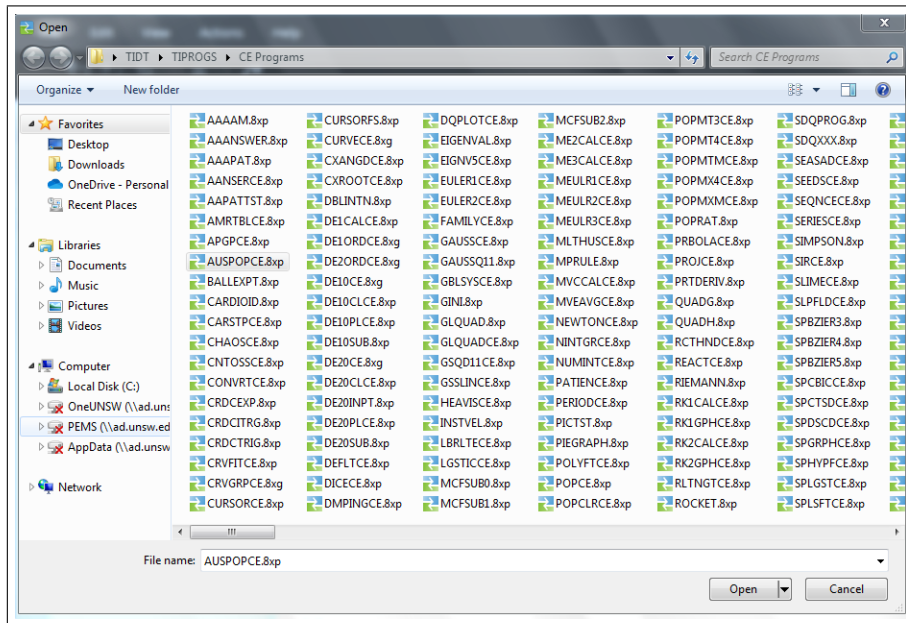
TI Connect
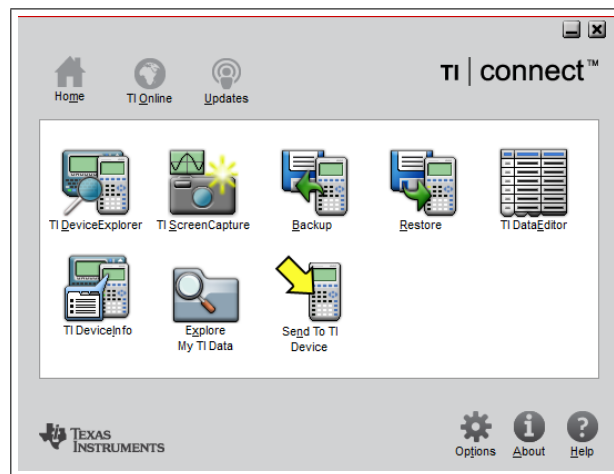
Click on *TI Device Explorer*. A window for your calculator should open, showing you what's on it (figure next page).

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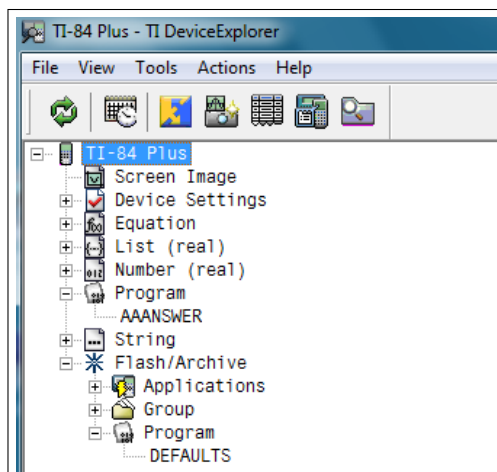
Click on the + sign next to *Program* to show the unarchived programs on the calculator. To see the archived programs (see Section 16.3), click on the + sign next to *Program* under *Flash/Archive*.

Drag the program from your desktop or folder where you have copied the program onto the calculator window, and the transfer should happen.

TI Connect will also work with the mini-USB cable and both calculator types but is slower and lacks the program editor of TI Connect CE.

16.2.2   From another calculator

Do this using the $\boxed{\text{link}}$ key $\left(\boxed{\text{2nd}}\ \boxed{\text{X,T,}\theta\text{,n}}\right)$. This gives a menu like that below left (TI-84CE).

The receiver moves the cursor to RECEIVE with the right arrow and presses $\boxed{\text{enter}}$ (below centre). This needs to be done before the sender transmits.
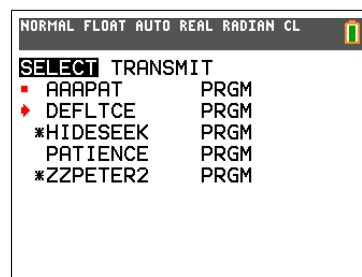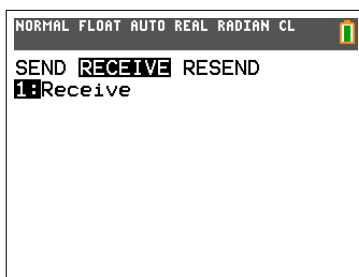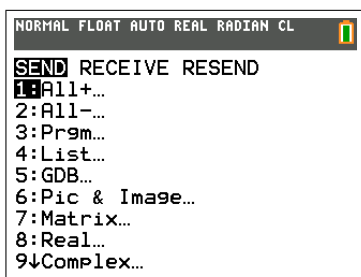
The sender presses $\boxed{3}$ (Prgm...), selects the program(s) to be sent with the cursor (above right) and $\boxed{\text{enter}}$, moves the cursor to TRANSMIT with the right arrow and presses $\boxed{\text{enter}}$ to start transmission.

If a program is already on the receiving calculator, you will get a menu of options.

Press $\boxed{\text{quit}}$ on both calculators to return to the Home screen.

16.2.3   Group program files

Sometimes, several programs are grouped together in a group file, for example when a program has several subroutines such as the DE20PTCE program; this program and its two subroutines are grouped together in the group file DE20CE.8xg, the g denoting a group file. This ensures that, when you copy the group file to your calculator, you have all the necessary programs.

```
NORMAL FLOAT AUTO REAL RADIAN CL

MEMORY
1:About
2:Mem Management/Delete…
3:Clear Entries
4:ClrAllLists
5:Archive
6:UnArchive
7:Reset…
8:Group…
```

```
NORMAL FLOAT AUTO REAL RADIAN CL

GROUP  UNGROUP
1:Create New
```

```
NORMAL FLOAT AUTO REAL RADIAN CL

GROUP  UNGROUP
1:*CURVECE
2:*DE20CE
```

To recover the individual programs, press ⬚mem⬚ (⬚2nd⬚ ⬚+⬚) (above left), scroll down to *Group...* and press ⬚enter⬚. Here you can create a new group file or ungroup an existing one (above centre). The latter is what we need here. Move the cursor to UNGROUP and press ⬚enter⬚. The group file you have just copied to your calculator should be listed here (above right). Press the number against its name and all should happen. If you already have some of the programs on your calculator, a menu will appear giving you options.

16.3   Archiving and unarchiving

Programs on your calculator can be either in main memory (RAM), and available to run and edit, or archived (ARC) and not available to run (except on the latest CE) or edit. Archived programs are indicated by an asterisk in the program list.

```
NORMAL FLOAT AUTO REAL RADIAN CL

MEMORY
1:About
2:Mem Management/Delete…
3:Clear Entries
4:ClrAllLists
5:Archive
6:UnArchive
7:Reset…
8:Group…
```

```
NORMAL FLOAT AUTO REAL RADIAN CL

RAM FREE        147544
ARC FREE         2914K
1:All…
2:Real…
3:Complex…
4:List…
5:Matrix…
6:Y-Vars…
7:Prgm…
8↓Pic & Image…
```

```
NORMAL FLOAT AUTO REAL RADIAN CL

RAM FREE        147544
ARC FREE         2914K
▶ AAAPAT           31
  DEFLTCE          73
 *HIDESEEK        117
  PATIENCE       3045
 *ZZPETER2         17
 *Pic6           21956
  L₁               48
  L₂               12
```

To unarchive (or archive) a program, press ⬚mem⬚ (⬚2nd⬚ ⬚+⬚) ⬚2⬚ (Memory Management) (above left) and ⬚1⬚ (All...) (above centre). You will see a list there of everything on your calculator, starting with the programs (above right). To unarchive a program, move the cursor onto it and press ⬚enter⬚; the asterisk should disappear. To archive it, press ⬚enter⬚, which just toggles between the two states.

Press ⬚quit⬚ to return to the Home screen.

16.4   Program protection

Programs in the PROTECTED folders have been locked or protected. This means they can be run and deleted but not altered in any way. Alteration usually happens when the user stops a program using $\boxed{\text{on}}$ but then (inadvertently) selects GoTo instead of Quit. Any subsequent key presses will alter the program, meaning it will not work as it should or at all. This is the greatest source of frustration in a class. The only cure is to copy the program again from someone else.

Using the protected version of the program means that this cannot happen. Recommended for use in class.

The downside of protecting a program is that the program cannot be read and perhaps modified on purpose. That is why both protected and unprotected programs are provided. The only way of knowing whether a program on your calculator is protected is to try to edit it; the calculator won't let you if it is.

Protecting and unprotecting a program is done in the editor of TI Connect CE (but not TI Connect). The editor is the bottom symbol in the column of three symbols on the left-hand side of the initial TI Connect CE screen. Once a program is displayed, there is a box to click on to protect/unprotect it.

16.5   Using the calculator — Some advice

- If your calculator stops during a program, try $\boxed{\text{enter}}$, $\boxed{\text{on}}$, $\boxed{\text{quit}}$, $\boxed{\text{off}}$ in that order. $\boxed{\text{enter}}$ restarts a program after a Pause (moving line of dots or small rotating circle in the top right-hand corner of the screen), while $\boxed{\text{on}}$ stops most programs running. If all else fails, you will have to remove the batteries and press $\boxed{\text{on}}$ a few times.

- Read the calculator guidebook on the order of evaluation of expressions. You will, at some time, end up with wrong answer if you are not aware of this. For example, $(^-1)^2 = 1$, but $^-1^2 = ^-1$, that is negation is done after raising to a power.

- Be wary of round-off error. Very small numbers such as 1_E^-13 (10^{-13}) are likely to be zero.

- If your TI-84Plus batteries are nearly flat, the screen will blank out when graphs are plotted or the contrast will need to be set to 8 or 9 and/or you will see a message on the screen when you switch on the calculator. Change the batteries one at a time. There is a back-up battery that is supposed to preserve all memory when you change the main batteries, but it doesn't always work.

  The CE has a rechargeable battery.

- If the TI-84Plus RAM memory is full (usually because you have lots of games in your calculator), the calculator will tell you. You will then need to erase something or copy it to Archival memory using the $\boxed{\text{mem}}$ menu.[61] Here, you can check how much memory is left, how much memory each program takes up and also delete programs, etc from the calculator memory (press $\boxed{\text{del}}$).

  The CE has masses of memory, both RAM and Archival, so filling up the memory is unlikely.

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[61]or to your computer using TI Connect/CE

- The best way to get a program is to copy it electronically from a reliable source. Check the examples given with most program instructions to see if your copy of the program is OK. Electrons occasionally go astray!

- If you do type in a program, take note of the following. Most program commands, functions and special variable names such as Y_1, Xmax, etc. are found in menus. You do not type them in letter by letter; if you do, the program will not run. *Anything containing lower-case letters can only be found in a menu.*

  A summary of the menus is in the Guidebook.

16.6   Programs

The programs here are available from *www.YYY*.

16.6.1   Activities for Years 9 and 10

Programs used in *Volume 1 Supplement: Activities for Years 9 and 10*. Details and examples are given there. Programs are in the ACTIVITIES910 folder.

CARSTOP/CARSTPCE — used in *Reaction Times and Statistics*.

GUESSLIN/GESLINCE — used in *Guess the Line*.

PARABOLA/PRBOLACE — used in *Parabolic Aerobics*.

REACT/REACTCE — used in *Reaction Times and Statistics*.

REACTHND/RCTHNDCE — used in *Reaction Times and Statistics*.

SLIME/SLIMECE — used in *Alien Attack*.

16.6.2   Fitting functions to data

Programs used in *Fitting Functions to Data 2* in Volume 3 of this book. Details and examples are given there. Programs are in the CURVEFIT folder.

CURVFIT/CRVFITCE — fits 19 different functions to a data set.
Subroutines **MCFSUB0**, **MCFSUB1** and **MCFSUB2**.
CURVE.8xg/CURVECE.8xg
Group file containing the main programs and 3 subroutines.

MOVEAVGE/MVEAVGCE — moving averages

POLYFIT/POLYFTCE — exact polynomial fit to data

SEASADJS/SEASADCE — seasonally adjusted values

16.6.3   Differential equations

Programs used in *Differential Equations*, Chapter 13. Details and examples are given there. Programs are in the DIFFEQNS folder.

SLPFIELD/SLPFLDCE
Plots slope fields for first-order DEs.

EULER1/EULER1CE
Uses Euler's method to plot approximate solutions to a first-order DE.

MODEULR1/MEULR1CE
Uses the modified Euler's method to plot approximate solutions to a first-order DE.

ME1CALC/ME1CALCE
Uses the modified Euler's method to calculate approximate solutions to a first-order DE.

RK1CALC/RK1CALCE
Uses a fourth-order Runge-Kutta method to calculate final values for a first-order DE.

RK1GRAPH/RK1GPHCE
Uses a fourth-order Runge-Kutta method to plot approximate solutions to a first-order DE.

DE1CALC/DE1CALCE

Compendium program containing all three methods.

DE1ORD.8xg/DE1ORDCE.8xg

Group file containing all the above first-order DE solvers.

EULER2/EULER2CE

Uses Euler's method to plot approximate solutions to two coupled first-order DEs or a second-order DE.

MODEULR2/MEULR2CE

Uses the modified Euler's method to plot approximate solutions to two coupled first-order DEs or a second-order DE.

ME2CALC/ME2CALCE

Uses the modified Euler's method to calculate final values for two coupled first-order DEs or a second-order DE.

RK2GRAPH/RK2GPHCE

Uses a fourth-order Runga-Kutta method to plot approximate solutions to two coupled first-order DEs or a second-order DE.

RK2CALC/RK2CALCE

Uses a fourth-order Runga-Kutta method to calculate final values for two coupled first-order DEs or a second-order DE.

DAMPING/DMPINGCE

Plots solutions for a damped harmonic oscillator (second-order DE) with different values for the damping coefficient.

SPSHMA/SPSHMACE

Plots the position and the velocity versus time of a particle undergoing simple harmonic motion; parameters can be changed on-screen.

SPSHMB/SPSHMBCE

Plots the actual path, the position and the velocity versus time of a particle undergoing simple harmonic motion; parameters can be changed on-screen.

DE2ORD.8xg/DE2ORDCE.8xg

Group file containing all the above second-order DE solvers.

MODEULR3/MEULR3CE

Uses the modified Euler's method to plot approximate solutions to three coupled first-order DEs or a third-order DE.

ME3CALC/ME3CALCE

Uses the modified Euler's method to calculate approximate solutions to three coupled first-order DEs.

DE20CALC/DE20CLCE

DE20PLOT/DE20PLCE

Calculates/plots the solutions of up to 20 coupled first-order DEs or of DEs of order up to 20. Choice of the three methods. Requires **DE20SUB** and, for more than 10 DEs, **DE20INPT**.

DE20.8xg/DE20CE.8xg

Group file containing all the DE20/DE20CE programs (4).

16.6.4   Laboratory programs

Used in Volume 4. Details of the programs are given in the relevant labs there.

Programs are in the LABPROGS folder.

COINTOSS/CNTOSSCE — simulates tossing a coin

Used in *Gambling Returns.*

FAMILY2/FAMLY2CE — plots families of curves.

Used in *Graphics and Calculus: Families of Curves.*

GAMBLSYS/GBLSYSCE — simulates five different gambling systems

Used in *Gambling Tactics.*

LABRLTTE/LBRLTECE — simulates a roulette wheel

Used in *Gambling Tactics.*

LOGISTIC/LGSTCE — discrete logistic model

Used in *Kangaroo Management.*

NEWTON/NEWTONCE — Newton's Method for finding zeros of functions

Used in *Newton's Method: A Classic Problem*

NINTGRPH/NINTGRCE — numerical integration

Used in *Rectangles, Area and the Definite Integral* and *Approximating Definite Integrals.*

NUMINT/NUMINTCE — numerical integration

Used in *Approximating Definite Integrals.*

PROJ/PROJCE — projecting an object over a wall

Used in *Projectile.*

STOPWTCH/STPWCHCE — timer for balloon drop

Used in *Balloons, Submarines and Drag.*

WALK/WALKCE — simulates a roulette wheel, betting on red

Used in *Gambling Tactics.*

WINBET/WINBETCE — simulates betting

Used in *Gambling Returns.*

16.6.5   Numerical integration

Use in *Numerical Integration*, Chapter 11. Details of the programs are given there.

The programs calculate approximate values for $\int_A^B f(X)\,dX$.

Programs are in the NINT folder.

NINTGRPH/NINTGRCE

Approximates the integral using the *Left-Endpoint Rule*, the *Right-Endpoint Rule*, the *Trapezoidal Rule*, the *Midpoint Rule* and *Simpson's Rule* with **N sub-divisions** (N+1 if N is odd, for Simpson's Rule), and draws the corresponding approximations to the function on each subinterval.

NUMINT/NUMINTCE

Approximates the integral using the *Left* (**L**) and *Right* (**R**) *Endpoint Rules*, the *Trapezoidal Rule* (**T**) and the *Midpoint Rule* (**M**), all with **N sub-divisions**, and *Simpson's Rule* (**S**) with **2N sub-divisions** (to ensure an even number of sub-intervals).

SPNUMINT/SPNINTCE

Spinner version (see Section 16.6.8) of NUMINT/NUMINTCE; the number of intervals N and the integration method M can be changed interactively with the arrow keys. They also calculate the approximation to the integral using the chosen method and the 'exact' integral using *fnInt*. The HELP option gives details of the methods and how the graphics work.

GLQUAD/GLQUADCE

Gauss-Legendre quadrature (integration) of orders 9 and 11.

GAUSSQ11/GSQD11CE

Gauss-Legendre quadrature (integration) and Gauss-Kronrod quadrature of order 11.

NINT.8xg/NINTCE.8xg

Group file containing all the numerical-integration programs.

16.6.6   Population modelling

Programs are in the POPN folder.

AUSPOP/AUSPOPCE — Australian population 1906 – 2016

Used in *Population Modelling 1* in Volume 1. Details there.

LOGISTIC/LGSTICCE — populations of bacteria or kangaroos

Used in *Population Modelling 2: Logistic and Epidemic Models*, Chapter 14. Details there.

SPLOGIST/SPLGSTCE are spinner versions (see Section 16.6.8) of LOGISTIC/LGSTICCE, allowing you to change the parameters on-screen and see the result immediately replotted.

MALTHUS/MLTHUSCE — US population 1790 – 2020

Used in *Population Modelling 1* in Volume 1. Details there.

POP/POPCE — population-matrix modelling

Used in *Population Modelling 3: Matrix Models*, Chapter 14.3. Details there.

POPMX4/POPMX4CE — population-matrix modelling

Used in *Population Modelling 3: Matrix Models*, Chapter 14.3. Details there.

POPMXM/POPMXMCE — population-matrix modelling

Used in *Population Modelling 3: Matrix Models*, Chapter 14.3. Details there.

POPCLEAR/POPCLRCE — resets defaults and deletes lists

Used in *Population Modelling 3: Matrix Models*, Chapter 14.3. Details there.

SIR/SIRCE — plots the results of the SIR model of an epidemic

Used in *Population Modelling 2: Logistic and Epidemic Models*, Chapter 14. Details there.

WHALES/WHALESCE — population-matrix modelling

Used in *Population Modelling 3: Matrix Models*, Chapter 14.3. Details there.

WORLDPOP/WLDPOPCE — World population 1940 – 2020 vspace-1mm

Used in *Population Modelling 1* in Volume 1. Details there.

16.6.7   Probability and Statistics

These programs are used in *Probability and Statistics 2* in Volume 3 of this book. Details of the programs and examples are given there. Programs are in the PROBSTATS folder.

COINTOSS/CNTOSSCE – simulates tossing a coin

DICE/DICECE – simulates tossing dice

GAMBLSYS/GBLSYSCE — simulates various gambling systems

LABRLTTE/LBRLTECE — simulates spins of a roulette wheel

ROULETTE (84+ only) — shows spin on a wheel

SPDISPDS/SPDSCDCE — discrete probability distributions (spinner)

SPCTSPDS/SPCTCDCE — continuous probability distributions (spinner)

SPLSQFIT/SPLSFTCE — allows you to fit a straight line to data (spinner)

WALK/WALKCE — simulates a random walk

WINBET/WINBETCE — simulates gambling games

16.6.8   Spinner programs

Calculator versions of some of the famous Spinners, characterised by being able to change parameters on the screen, usually with sliders. The calculator versions are not nearly as fancy, limited mostly by a much smaller screen, but they do allow parameters to be changed easily while looking at the graphics screen — interactive graphing. The HELP option in each program tells you how to change the parameters.

The spinners below are used in *Functions and their Graphs*, Chapter 9. Details of the programs and examples are given there. Programs are in the SPINNERS folder.

SPCPQUAD/SPQUADCE — roots of a quadratic equation

SPCUBIC/SPCBICCE — graphs a cubic equation

SPQUART/SPQRTCE — graphs a quartic equation

SPQUINT/SPQNTCE — graphs a quintic equation

SPGRAPH/SPGRPHCE — graphs a function with 3 parameters

SPSUMSIN/SPSSINCE — addition of two sine functions

SPTRIGF/SPTRIGCE — graphs trig functions

SPHYPBFN/SPHYPFCE — graphs hyperbolic functions

SPPWEXLN/SPPELCE — graphs power, exponential and log functions

SPBZIER3, **SPBZIER4**, **SPBZIER5** (CE only) — Bezier curves with 2, 3 and 4 control points, respectively.

16.6.9   Other programs

Programs are in the OTHERPROGS folder.

AAANSWER/AANSERCE — converts decimals to surds or multiples of π
Many problems done by hand (and answers in texts) involve surds or multiples of π whereas the calculator only works with decimal numbers. This program checks if a decimal number is a surd (mutiple of $\sqrt{2}$ or $\sqrt{3}$, or the square or cube root of some number) or a multiple of π. After you have generated the decimal number, just run the program.

AMORTTBL/AMRTBLCE — amortisation tables
Used in *Financial Mathematics 2* in Volume 3 of this book. Details are given there.

APGP/APGPCE — sums of APs and GPs
Used in *Sequences and Series* in Volume 3 of this book. Details are given there.

CHAOS/CHAOSCE — logistic sequence
Used in *Sequences and Series* in Volume 3 of this book. Details are given there.

CMPXANGD/CPXANGCE — correct display of complex numbers
Used in *Complex Numbers* in Volume 3 of this book. Details are given there.

CMPXROOT/CXROOTCE — roots of complex numbers
Used in *Complex Numbers* in Volume 3 of this book. Details are given there.

DIFFQPLT/DQPLOTCE — local linearity
Used in *Graph and Calculus Operations*, Chapter 10. Details are given there.

EIGENV5/EIGENV5CE — eigenvalues and eigenvectors of a matrix
Used in *Matrix and Vector Operations* in Volume 3 of this book. Details are given there.

GAUSS/GAUSSCE — Gauss and Gauss-Jordan reduction
Used in *Matrix and Vector Operations* in Volume 3 of this book. Details are given there.

HEAVISDE/HEAVISCE — Heaviside step function
Used in *Functions and their Graphs*, Chapter 9. Details and examples are given there.

MVCCALC/MVCCALCE — multi-variable Calculus operations
Used in *Multi-variable Calculus*, Chapter 15. Details are given there.

PERIODIC/PERIODCE — draws periodic functions
Used in *Functions and their Graphs*, Chapter 9. Details and examples are given there.

SEQUENCE/SEQNCECE — limit of a sequence
Used in *Series and Sequences* in Volume 3 of this book. Details are given there.

SERIES/SERIESCE — sum a series

Used in *Series and Sequences* in Volume 3 of this book. Details are given there.

TAYLOR1/TAYLR1CE — Plots Taylor Series

Used in *Taylor Series*, Chapter 12. Details are given there.

TAYLOR2/TAYLR2CE — Plots Taylor Series

Used in *Taylor Series*, Chapter 12. Details are given there.

VECTOR/VECTORCE — vector operations

Used in *Matrix and Vector Operations* in Volume 3 of this book. Details are given there.