



Specialist Unit 3 Revision

Complex Numbers

Simple Familiar

Question 1.

Evaluate the following: $i + i^2 + i^3 + i^4 + \dots + i^{99} + i^{100} + i^{101}$

- a) 0 b) -1 c) i d) $-i$ e) 1

Question 2.

Factorise the following quadratic into linear factors over \mathbb{C} : $z^2 + 36$

- a) $(z - 36)(z + 36)$ b) $(z - 6)(z + 6)$
 c) $(z - 6i)^2$ d) $(z + 6i)^2$
 e) $(z - 6i)(z + 6i)$

Question 3.

Solve the following equation for z : $2z^2 = 4z - 10$

- a) $-1 + 2i, -1 - 2i$ b) $-1 + 2i, 1 + 2i$
 c) $1 - 2i, 1 + 2i$ d) $-2 - i, -2 + i$
 e) $2 - i, 2 + i$

Question 4.

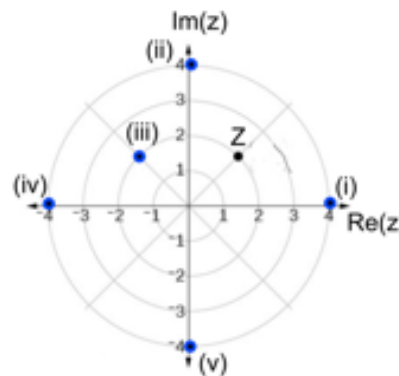
Find all solutions of the equation $z^4 + 8z^2 - 9 = 0, z \in \mathbb{C}$.

- a) $\pm 9i, \pm 1$ b) $\pm 3i, \pm 1$ c) $-9i, 1$ d) $3i, -1$ e) $3i, 1$

Question 5.

Given that $z = \sqrt{2} + \sqrt{2}i$, which one of the following points on the Argand diagram represents z^2 ?

- a) (i) b) (ii)
 c) (iii) d) (iv)
 e) (v)



Question 6.

Given $P(z) = 2z^3 + 8z^2 - 20z + 24 = 0$, $z \in \mathbb{C}$. Which one of the following is a linear factor of $P(z)$?

- a) $z + 1 + i$ b) $z - 1 + i$ c) $z + 1 - i$ d) $-z - 1 - i$ e) $-z - 1 + i$

Question 7.

Solve for z : $z^3 - 3z^2 + 5z - 3 = 0$, $z \in \mathbb{C}$

- a) $1 - \sqrt{2}i$, $1 + \sqrt{2}i$, -1 b) $-1 - \sqrt{2}i$, $1 + \sqrt{2}i$, -1
 c) $-1 - \sqrt{2}i$, $-1 + \sqrt{2}i$, -1 d) $1 - \sqrt{2}i$, $1 + \sqrt{2}i$, 1
 e) $-1 - \sqrt{2}i$, $1 + \sqrt{2}i$, 1

Complex Familiar**Question 8.**

The complex numbers z_1 and z_2 are given by:

$$z_1 = x + 3i \text{ and } z_2 = 3 - 2i \text{ where } x \in \mathbb{Z}.$$

Determine the possible values of x , given $\left| \frac{z_1}{z_2} \right| = 1$.

- a) $x = \pm 1$ b) $x = 1 \text{ and } 2$ c) $x = \pm 2$ d) $x = \pm\sqrt{2}$ e) $x = \pm i$

Question 9.

What is the sum of the complex roots of unity for the polynomial? $z^4 = 1$

- a) $2i$ b) $2 - 2i$ c) $2 + 2i$ d) 2 e) 0

Question 10.

Which one of the following represents the sum and product of the roots for the polynomial?

$$P(z) = z^5 + z^4 + z^3 + z^2 + z + 1$$

- a) 0 and 1 b) -1 and -1 c) -1 and 1 d) 1 and -1 e) 0 and 0

Complex Unfamiliar**Question 11.**

For the polynomial $P(z) = z^4 - 4z^3 + 7z^2 - 4z + 6$, $z = i$ is a solution of $P(z) = 0$. Determine a quadratic factor of $P(z)$ in the form of $z^2 + bz + c$.

- a) $z^2 + 1$ b) $z^2 - 1$
 c) $z^2 - 4z - 6$ d) $z^2 - 4z + 6$
 e) $z^2 + 4z - 6$

Question 12.

Let $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$. Find the exact value of: $z^1 \times z^2 \times z^3 \times z^4 \times \dots \times z^{98} \times z^{99} \times z^{100}$

- a) 1 b) -1 c) $-i$ d) i e) 0

Question 13.

Given $z = (1 + i)^n$, what value of n satisfies the following equation? $|z| = 16$

- a) $n = 4$ b) $n = 5$ c) $n = 6$ d) $n = 7$ e) $n = 8$

Answers

1. C	2. E	3. C	4. B	5. B	6. A	7. D	8. C	9. E	10. B	11. D
12. D	13. E									

Question 1. Answer C

Each sum of 4 terms results in a total of 0.
Since 101 has a remainder of 1 when divided by 4. The sum will be equal to i^1 or i .

Alternatively using sigma notation results in a sum of i .

NORMAL FLOAT AUTO a+bi RADIAN MP

$$\sum_{N=1}^{101} (i^N)$$

4.9E-12+i

Note: The real part of the answer is 4.9×10^{-12} which is approximately zero, therefore the answer is simply: i

Question 2. Answer E

Can factorise by hand, in this simple example it will most likely be quicker.

NORMAL FLOAT DEC a+bi DEGREE MP
PLYSMLT2 APP

$$1x^2 + 0x + 36 = 0$$

x1 = 6i
x2 = -6i

MAIN MODE COEFFSTORE F<D>

Results obtained using the PLYSMLT2 Application.
These are the roots and need to be written as factors.

Question 3. Answer C

$$2z^2 = 4z - 10 \text{ (rearrange)}$$

$$2z^2 - 4z + 10 = 0$$

Use quadratic formula or factorise:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{(-4)^2 - 4 \times 2 \times 10}}{2 \times 2}$$

$$= \frac{4 \pm \sqrt{-64}}{4}$$

$$= \frac{4 \pm 8i}{4}$$

$$= 1 \pm 2i$$

NORMAL FLOAT AUTO a+bi DEGREE MP
PLYSMLT2 APP

$$2x^2 - 4x + 10 = 0$$

x1 = 1+2i
x2 = 1-2i

MAIN MODE COEFFSTORE F<D>

Results obtained using the PLYSMLT2 Application.

Question 4. Answer B

$$z^4 + 8z^2 - 9 = 0$$

$$\text{Let } y = z^2$$

$$y^2 + 8y - 9 = 0$$

$$(y + 9)(y - 1) = 0$$

$$y = -9, y = 1$$

$$\text{Therefore } z^2 = -9 \text{ and } z^2 = 1$$

$$z = \pm 3i \text{ and } z = \pm 1$$

```

NORMAL FLOAT DEC a+bi DEGREE MP
PLYSMLT2 APP
a4x^4+a3x^3+a2x^2+a1x+a0=0
x1=3i
x2=-3i
x3=1
x4=-1
MAIN MODE COEFF STORE
    
```

Results obtained using the PLYSMLT2 Application. Be careful with coefficient entry.

Question 5. Answer B

$$z = \sqrt{2} + \sqrt{2}i$$

$$z^2 = (\sqrt{2} + \sqrt{2}i)^2$$

$$= (\sqrt{2} + \sqrt{2}i)(\sqrt{2} + \sqrt{2}i) = \sqrt{2} \times \sqrt{2} + \sqrt{2} \times \sqrt{2}i + \sqrt{2} \times \sqrt{2}i + \sqrt{2}i \times \sqrt{2}i$$

$$= 2 + 4i - 2$$

$$= 4i$$

```

NORMAL FLOAT AUTO a+bi RADIAN MP
PLYSMLT2 APP
(sqrt(2)+sqrt(2)i)^2
4i
MAIN MODE COEFF STORE
    
```

Numerical result only, while it is possible to graph the results, for something as 'trivial' as this that is not necessary or appropriate.

Question 6. Answer A

There are two ways of doing this. First by using the **Plysm1t2 Application**. Therefore, the factors of $P(z)$ are $(z + 6)$, $(z - 1 + i)$ and $(z - 1 - i)$.

Therefore $(z - 1 - i)$ is the correct answer.

Or by defining the polynomial $P(z)$ and testing which multi-choice answer results in a zero for $P(z)$.

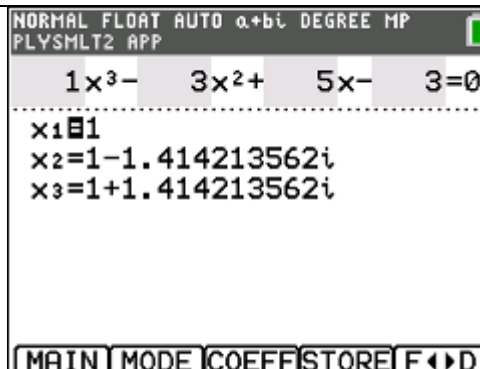
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NORMAL FLOAT AUTO a+bi DEGREE MP
PLYSMLT2 APP
2x^3+ 8x^2- 20x+ 24=0
x1=-6
x2=1-i
x3=1+i
MAIN MODE COEFF STORE F<D
    
```

PLYSMLT2 will be able to obtain solutions and will then need to convert into factors.

Question 7. Answer D

As per Question 6 – There are two ways to solve the problem. Check factors provided in multiple choice responses or use the PLYSMLT2 Application.



Note – While the approximate values for the irrational solutions are displayed, certain irrational numbers should be recognisable including $\sqrt{2}$, $\sqrt{3}$, π and e . and simple multiples of these.

Question 8. Answer C

$\left| \frac{x + 3i}{3 - 2i} \right| = 1$
To divide by a complex number, multiply by the conjugate.

$$\frac{(x + 3i)(3 + 2i)}{(3 - 2i)(3 + 2i)} = \frac{3x + 2ix + 9i - 6}{13}$$

$$= \frac{3x - 6 + (2x + 9)i}{13}$$

$$= \frac{3x-6}{13} + \frac{2x+9}{13}i$$

$$\left| \frac{3x-6}{13} + \frac{2x+9}{13}i \right| = \sqrt{\left(\frac{3x-6}{13}\right)^2 + \left(\frac{2x+9}{13}\right)^2}$$

$$= \sqrt{\frac{9x^2 - 36x + 36 + 4x^2 + 36x + 81}{169}}$$

$$= \sqrt{\frac{13x^2 + 117}{169}}$$

Now $\sqrt{\frac{13x^2 + 117}{169}} = 1$

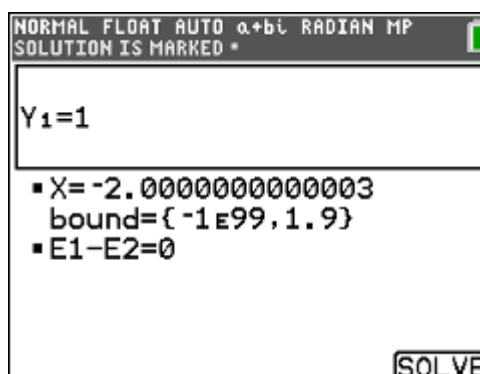
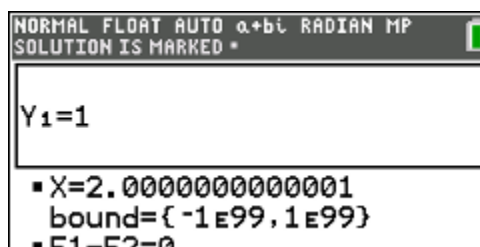
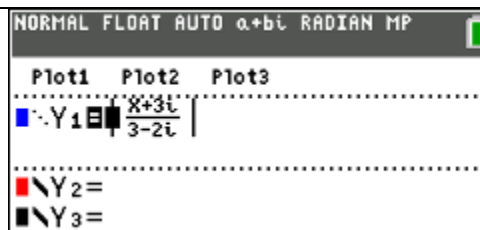
$$\frac{13x^2 + 117}{169} = 1$$

$$13x^2 + 117 = 169$$

$$13x^2 = 52$$

$$x^2 = 4$$

$$x = \pm 2$$



The y= editor is a convenient location to enter the equation. The numerical solver in the mathematics menu can then be used to determine solutions. Remember that only one solution is generated at a time. However, given this is a multiple choice question, the responses indicate that there are two solutions so restricting the domain or changing the initial guess will help find other solution(s).

Question 9. Answer E

$$z^4 = 1$$

Rearrange to: $z^4 - 1 = 0$

This is a polynomial with all real coefficients, therefore roots will appear as conjugate pairs, the sum of which must certainly produce no imaginary component.

```
NORMAL FLOAT DEC a+bi DEGREE MP
PLYSMLT2 APP
a4x^4+a3x^3+a2x^2+a1x+a0=0
x1=-1
x2=1i
x3=-1i
x4=1
MAIN MODE COEFFSTORE
```

```
NORMAL FLOAT AUTO a+bi RADIAN MP
sum(LA)
.....0
```

It should be easy to add the solutions without using a calculator, however the solutions can be stored to a list. In this example they have been stored to list "A" and then summed to get a result of zero.

Question 10. Answer B

Once again it should be apparent that the real coefficients in the original polynomial will produce conjugate roots.

In this example it is very useful to store the roots in a list and use the tools available in the List – Math menu to find both the sum and the product of the roots.

Answer is -1 and -1

```
NORMAL FLOAT DEC a+bi DEGREE MP
PLYSMLT2 APP
a5x^5+...+a1x+a0=0
x1=0.5+0.8660254038i
x2=0.5-0.8660254038i
x3=-1
x4=-0.5+0.8660254038i
x5=-0.5-0.8660254038i
MAIN MODE COEFFSTORE
```

```
NORMAL FLOAT AUTO a+bi RADIAN MP
sum(LB)
.....-1
Prod(LB)
.....-1
```

Question 11. Answer D

By hand:

Given $z = i$ is a solution and all coefficients are real, the conjugate is a solution also: $z = -i$, therefore $z^2 + 1$ is a factor. By progressive factorisation:

$$z^4 - 4z^3 + 7z^2 - 4z + 6$$

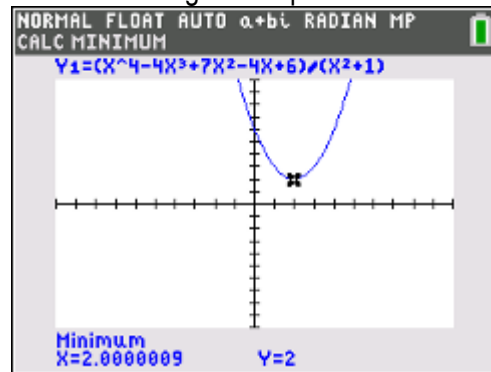
$$z^2(z^2 + 1) - 4z^3 + 6z^2 - 4z + 6$$

$$z^2(z^2 + 1) - 4z(z^2 + 1) + 6z^2 + 6$$

$$z^2(z^2 + 1) - 4z(z^2 + 1) + 6(z^2 + 1)$$

$$(z^2 + 1)(z^2 - 4z + 6)$$

The PLYSMLT2 can be used to find the roots, however a graph can be just as useful, particularly since the leading coefficient of the most significant power is 1.



Dividing the original polynomial by ' $z^2 + 1$ ' will produce a quadratic. The turning point of the quadratic can be found by calculating the minimum.

The result is therefore: $(z + 2)^2 + 2 = z^2 + z + 6$

[Note the calculator uses x as the variable for graphs]

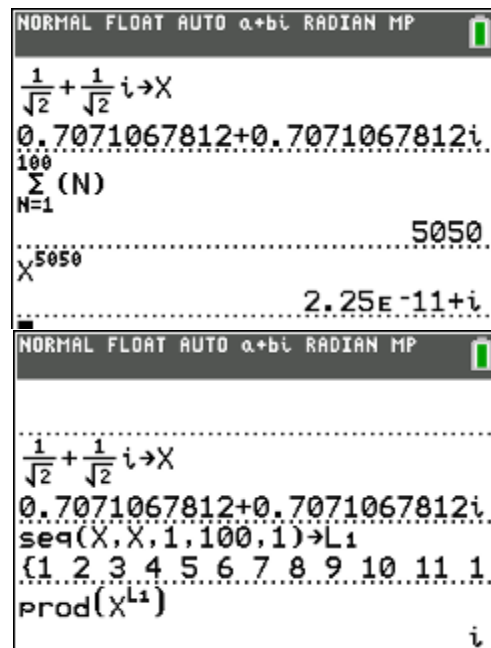
Question 12. Answer D

There are several ways to do this calculation.

Adding the powers $1 + 2 + 3 + \dots + 99 + 100$ gives 5050. The Sigma template can be used to determine the sum is 5050.

$$\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{5050} = i$$

Or using the product tool (generate a list of values first)

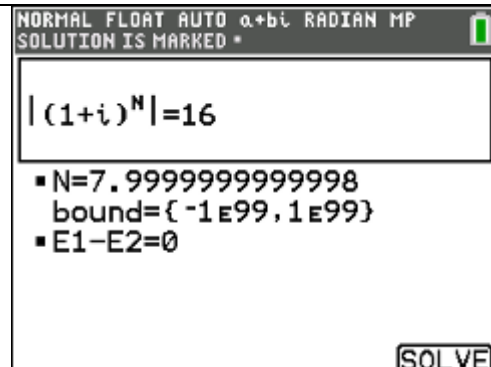


Question 13. Answer E

By Hand:

As $|z| = \sqrt{2}$ then it is relatively straight forward to extend this to $|z|^2 = 2$ and since $2^4 = 16$, then it follows that $|z|^{2 \times 4} = 16$ therefore $n = 8$.

The numerical solve can also be used on the calculator.



Question 6.

Find a unit vector that is perpendicular to both \mathbf{a} and \mathbf{b} .

$$\mathbf{a} = 3\mathbf{i} + 6\mathbf{j} + \mathbf{k}$$

$$\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$$

- a) $\frac{1}{\sqrt{2045}}(32\mathbf{i} + 11\mathbf{j} - 30\mathbf{k})$ b) $\frac{1}{\sqrt{2045}}(32\mathbf{i} + 11\mathbf{j} - 30\mathbf{k})$
- c) $\frac{1}{\sqrt{2045}}(32\mathbf{i} - 11\mathbf{j} + 30\mathbf{k})$ d) $\frac{1}{\sqrt{2045}}(32\mathbf{i} - 11\mathbf{j} - 30\mathbf{k})$
- e) $\frac{1}{\sqrt{2045}}(32\mathbf{i} - 11\mathbf{j} - 30\mathbf{k})$

Question 7.

The motion of two particles is modelled by the vector functions $\mathbf{r}_1(t) = (2t - 4)\mathbf{i} + (12 - t^2)\mathbf{j}$ and $\mathbf{r}_2(t) = (2 - t)\mathbf{i} + 4t\mathbf{j}$ where $t \geq 0$. Find the point at which the particles collide.

- a) (8, 0) b) (0, 8) c) (2, 8) d) (2, 0) e) (0, 2)

Question 8.

The motion of two particles is modelled by the vector functions $\mathbf{r}_1(t) = (2t - 1)\mathbf{i} + (t^2 + 10)\mathbf{j}$ and $\mathbf{r}_2(t) = (t + 2)\mathbf{i} + 8t\mathbf{j}$ where $t \geq 0$. Find the distance between the two particles when $t = 2$.

- a) 5 b) $\sqrt{5}$ c) $\sqrt{7}$ d) $\sqrt{30}$ e) $\sqrt{949}$

Complex familiar**Question 9.**

Consider the plane containing the points A (1, 2, 3), B (2, 4, 5) and C (-2, 1, 3).

Which one of the following represents a Cartesian equation of the plane?

- a) $z = \frac{5 - 2x + 6y}{5}$ b) $z = \frac{5 - 2x - 6y}{5}$
- c) $z = \frac{5 - 2x + 6y}{-5}$ d) $z = \frac{5 + 2x + 6y}{5}$
- e) $z = \frac{5 - 2x - 6y}{-5}$

Questions 10 and 11 refer to the following information.

A soccer ball is kicked from a starting height of 1 metre at a speed of 25 m/sec at an angle of 45° with respect to the ground.

Question 10.

Find the maximum height (h) reached by the soccer ball and the time (t) at which it occurs.

- a) $t = 1.8 \text{ sec}, h = 31.8 \text{ m}$ b) $t = 2 \text{ sec}, h = 16.9 \text{ m}$
- c) $t = 1.8 \text{ sec}, h = 16.9 \text{ m}$ d) $t = 2 \text{ sec}, h = 31.8 \text{ m}$
- e) $t = 2.1 \text{ sec}, h = 16.9 \text{ m}$

Answers

1. B	2. C	3. D	4. C	5. A	6. E	7. B	8. B	9. A	10. C	11. D	12. D
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Question 1. Answer B

$$x = 4 \cos\left(\frac{\pi t}{2}\right) \text{ and } y = 3 \sin\left(\frac{\pi t}{2}\right)$$

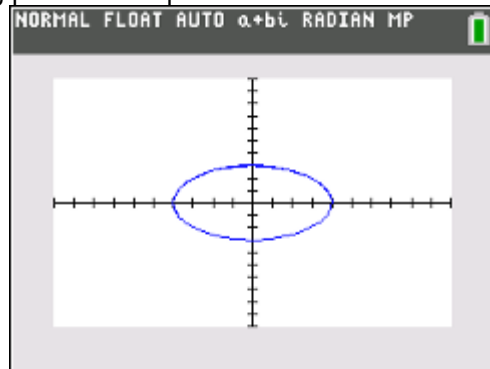
$$\cos\left(\frac{\pi t}{2}\right) = \frac{x}{4} \text{ and } \sin\left(\frac{\pi t}{2}\right) = \frac{y}{3}$$

$$\text{since } \cos^2\left(\frac{\pi t}{2}\right) + \sin^2\left(\frac{\pi t}{2}\right) = 1$$

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$$

Using parametric equations



The conics Application can be used for the Cartesian representation to confirm they are the same.

Question 2. Answer C

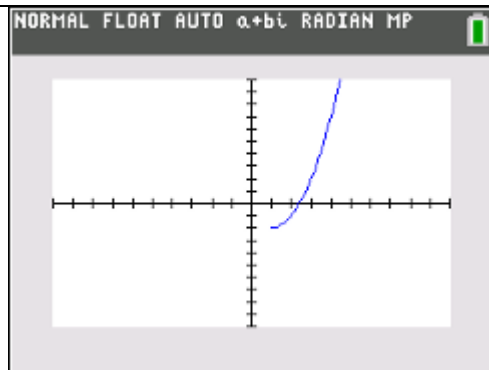
$$x = t + 1$$

$$t = x - 1$$

$$y = t^2 - 2$$

$$\text{therefore } y = (x - 1)^2 - 2$$

Note: Students may be confused by the domain and should expect that in an examination a domain restriction may also be included in the responses.



The parametric equations can be graphed with resultant graph, including the turning point location, producing: $y = (x - 1)^2 - 2$

Question 3. Answer D

$$x = -2 \cos\left(\frac{\pi t}{2}\right) \text{ and } y = 4 \sin\left(\frac{\pi t}{2}\right)$$

$$\cos\left(\frac{\pi t}{2}\right) = \frac{x}{-2} \text{ and } \sin\left(\frac{\pi t}{2}\right) = \frac{y}{4}$$

$$\text{since } \cos^2\left(\frac{\pi t}{2}\right) + \sin^2\left(\frac{\pi t}{2}\right) = 1$$

$$\left(\frac{x}{-2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

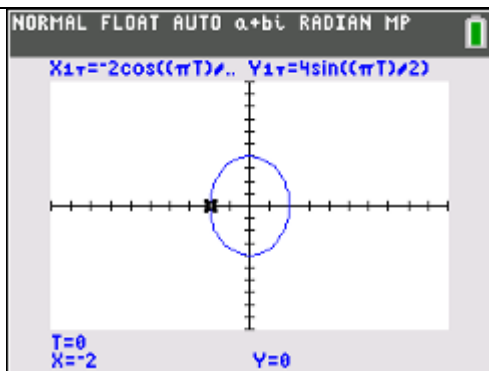
$$\frac{x^2}{2^2} + \frac{y^2}{4^2} = 1$$

When $t = 0$

$$r(0) = -2 \cos\left(\frac{\pi \times 0}{2}\right) i + 4 \sin\left(\frac{\pi \times 0}{2}\right) j$$

$$= -2j$$

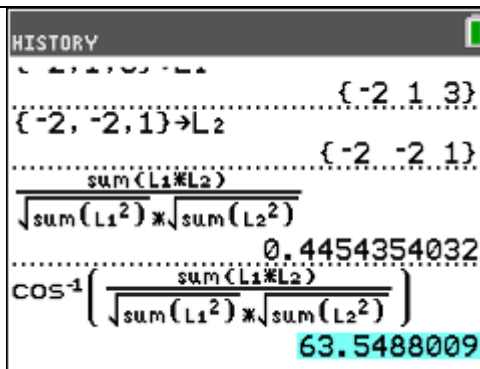
$$\text{The period is } 2\pi \div \frac{\pi}{2} = 4$$



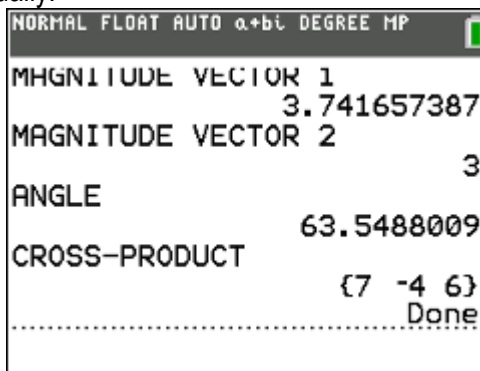
Using **Trace** with $T = 0$ produces starting point. The window settings, specifically 't-max' can be used to confirm the period = 4.

Question 4. Answer C

$$\begin{aligned} \cos(\theta) &= \frac{a \cdot b}{|a||b|} \\ &= \frac{-2 \times -2 + 1 \times -2 + 3 \times 1}{\sqrt{(-2)^2 + 1^2 + 3^2} \sqrt{(-2)^2 + (-2)^2 + 1^2}} \\ &= \frac{5}{11.225} \\ \theta &= \cos^{-1}\left(\frac{5}{11.225}\right) \\ &= 63.598^\circ \end{aligned}$$

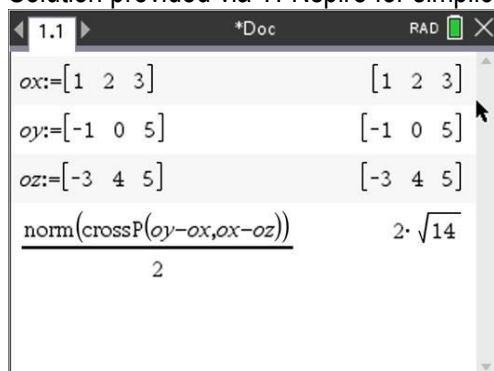


Vectors are not native to the TI-84PlusCE. It is possible to place the coefficients into a list and then use list commands to perform the calculations. A more efficient approach would be to write a small program (see below) ... or simply do the calculations manually.



Question 5. Answer A

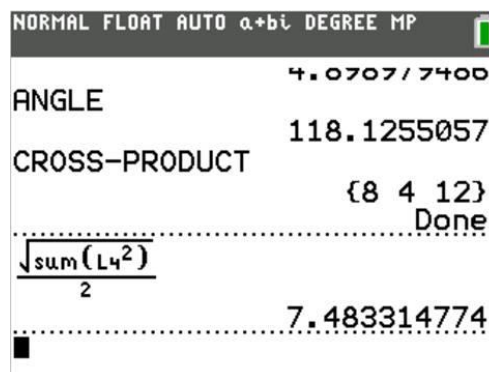
Using the cross product.
Solution provided via TI-Nspire for simplicity.



Whilst this question can be solved using the calculator's built in functionality, it would take some time!

An alternative to using the cross-product would be to calculate the magnitudes of the sides and then use Heron's formula.

The screen below shows the result of a simple Vector program that produces the cross-product and stores the result in L4.



Question 6. Answer E

Using the cross product.

$$\mathbf{a} = 3\mathbf{i} + 6\mathbf{j} + \mathbf{k}$$

$$\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 6 & 1 \\ 4 & -2 & 5 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = 32\mathbf{i} - 11\mathbf{j} - 30\mathbf{k}$$

Need the magnitude of the vector $\mathbf{a} \times \mathbf{b}$ in order to get the unit vector.

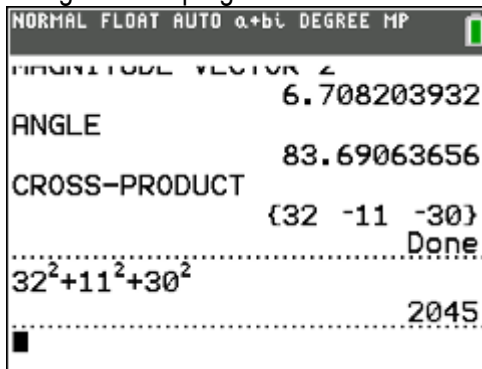
$$|\mathbf{a} \times \mathbf{b}| = |32\mathbf{i} - 11\mathbf{j} - 30\mathbf{k}|$$

$$= \sqrt{32^2 + (-11)^2 + (-30)^2}$$

$$= \sqrt{2045}$$

$$\text{Unit vector is } \frac{1}{\sqrt{2045}}(32\mathbf{i} - 11\mathbf{j} - 30\mathbf{k})$$

Using a vector program:



The cross-product is generated using the program. The unit vector for this cross-product can be determined manually or by using the result. To see the exact value, do the calculation without determining the *rational approximation* for the result:

$$\frac{1}{\sqrt{2045}}(32\mathbf{i} - 11\mathbf{j} - 30\mathbf{k})$$

Question 7. Answer B

$$\mathbf{r}_1(t) = (2t - 4)\mathbf{i} + (12 - t^2)\mathbf{j} \text{ and}$$

$$\mathbf{r}_2(t) = (2 - t)\mathbf{i} + 4t\mathbf{j}$$

Equating \mathbf{i} components.

$$2t - 4 = 2 - t$$

$$3t = 6$$

$$t = 2$$

Equating \mathbf{j} components.

$$(12 - t^2) = 4t$$

Use $t = 2$ to check

$$12 - 2^2 = 4 \times 2$$

$$8 = 8$$

So particles are together at $t = 2$.

$$\mathbf{r}_1(t) = (2t - 4)\mathbf{i} + (12 - t^2)\mathbf{j}$$

$$\mathbf{r}_1(2) = (2 \times 2 - 4)\mathbf{i} + (12 - 2^2)\mathbf{j}$$

$$\mathbf{r}_1(2) = (0)\mathbf{i} + (8)\mathbf{j} = 8\mathbf{j}$$

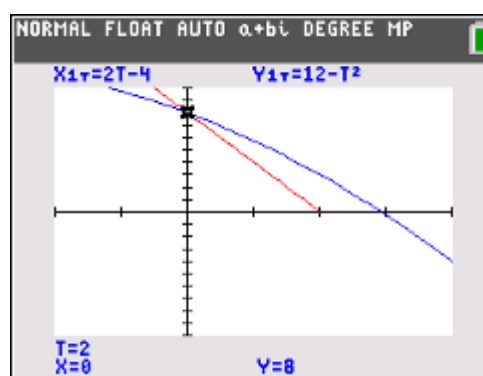
$$\mathbf{r}_2(t) = (2 - t)\mathbf{i} + 4t\mathbf{j}$$

$$\mathbf{r}_2(2) = (2 - 2)\mathbf{i} + 4 \times 2\mathbf{j}$$

$$\mathbf{r}_2(2) = 0\mathbf{i} + 8\mathbf{j} = 8\mathbf{j}$$

Coordinates are $(0, 8)$ at $t = 2$

Graphing the two equations in parametric



The vectors equations can be plotted using the parametric function tool. A small ΔT will make the point of intersection obvious. Alternatively the trace function can allow users to check the solution location. (see above).

As can be seen from the trace: $T = 0$, $X = 0$ and $Y = 8$.

Question 8. Answer B

$$\mathbf{r}_1(t) = (2t - 1)\mathbf{i} + (t^2 + 10)\mathbf{j}$$

$$\mathbf{r}_2(t) = (t + 2)\mathbf{i} + 8t\mathbf{j}$$

$$\mathbf{r}_1(2) = (2 \times 2 - 1)\mathbf{i} + (2^2 + 10)\mathbf{j} = 3\mathbf{i} + 14\mathbf{j}$$

$$\mathbf{r}_2(2) = (2 + 2)\mathbf{i} + 8 \times 2\mathbf{j} = 4\mathbf{i} + 16\mathbf{j}$$

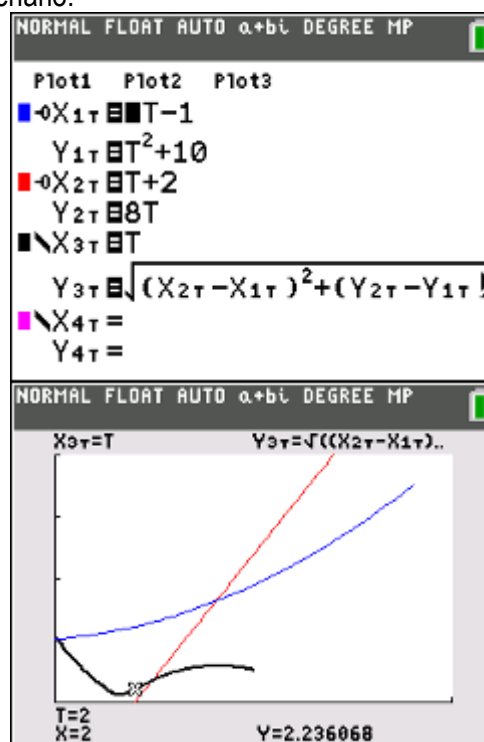
$$\mathbf{r}_2(2) - \mathbf{r}_1(2) = (4\mathbf{i} + 16\mathbf{j}) - (3\mathbf{i} + 14\mathbf{j})$$

$$\mathbf{r}_2(2) - \mathbf{r}_1(2) = \mathbf{i} + 2\mathbf{j}$$

$$|\mathbf{r}_2(2) - \mathbf{r}_1(2)| = \sqrt{1^2 + 2^2}$$

$$= \sqrt{5}$$

Some questions are quicker 'by hand'. However, if the 'distance between two vectors' is stored as a GDB then the calculator is reasonably efficient and provides a good visual of the scenario.



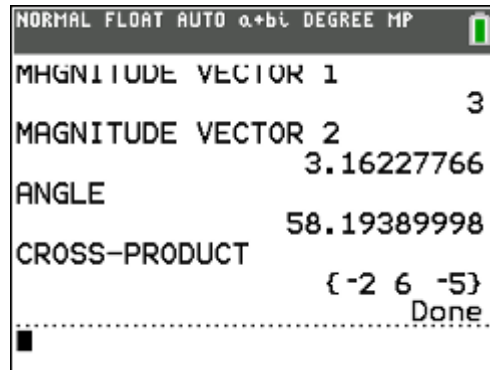
Question 9. Answer A

Using $[2, -6, 5]$ and the dot product

$$2x - 6y + 5z = 5$$

$$z = \frac{5 - 2x + 6y}{5}$$

To save time, using the vector program the vectors entered for V_1 and V_2 are $B - A$ and $A - C$:



The equation to the plane is given by:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Where a , b & c are from the cross product: $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and x_0 , y_0 and z_0 are the components from any one of the vectors A , B or C : $x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$

By hand, solve for z :

$$-2(x - 1) + 6(y - 2) - 5(z - 3) = 0$$

Note: In the above solution vector A has been used, but the same result is obtained if B or C is used.

$$z = \frac{5 - 2x + 6y}{5}$$

Question 10. Answer C

$$\mathbf{r}(t) = \left(25 \cos\left(\frac{\pi}{4}\right)t\right)\mathbf{i} + \left(25 \sin\left(\frac{\pi}{4}\right)t - \frac{1}{2} \times 9.8t^2 + 1\right)\mathbf{j}$$

For the vertical component:

$$y = 25 \sin\left(\frac{\pi}{4}\right)t - 4.9t^2 + 1$$

Maximum height occurs when $\frac{dy}{dt} = 0$.

$$\frac{dy}{dt} = 25 \sin\left(\frac{\pi}{4}\right) - 9.8t$$

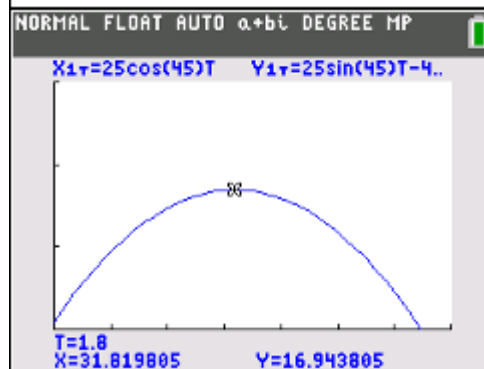
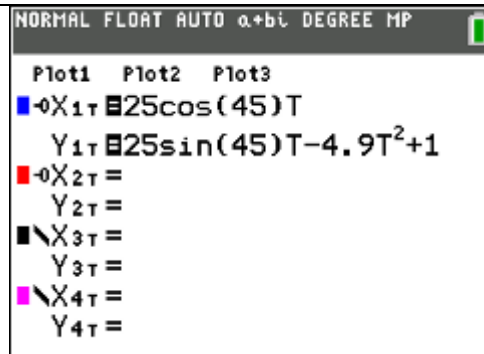
$$0 = 25 \sin\left(\frac{\pi}{4}\right) - 9.8t$$

$$25 \sin\left(\frac{\pi}{4}\right) = 9.8t$$

$$t = \frac{25 \sin\left(\frac{\pi}{4}\right)}{9.8}$$

$$t = 1.80384 \text{ sec}$$

$$y = 25 \sin\left(\frac{\pi}{4}\right) \times 1.8 - 4.9 \times 1.8^2 + 1 = 16.9 \text{ m}$$



Note that calculate > maximum is not available in parametric mode, but it is multiple choice and quick to substituted values for T .

Maximum height of 16.9 m at 1.8 seconds.

Question 11. Answer E

$$x = \left(25 \cos\left(\frac{\pi}{4}\right) t \right)$$

$$60 = \left(25 \cos\left(\frac{\pi}{4}\right) t \right)$$

$$t = \frac{60}{25 \cos\left(\frac{\pi}{4}\right)}$$

$$= 3.394 \text{ s [Rounding Warning]}$$

$$\text{Height} = 25 \sin\left(\frac{\pi}{4}\right) \times 3.394 - 4.9 \times (3.394)^2 + 1$$

$$= 4.54 \text{ m [Rounding Error]}$$

When approximate values are used repeatedly in a calculation the errors can be compounding. Note that in this example $t \approx 3.394$. The exact answer is $\frac{12\sqrt{2}}{5}$

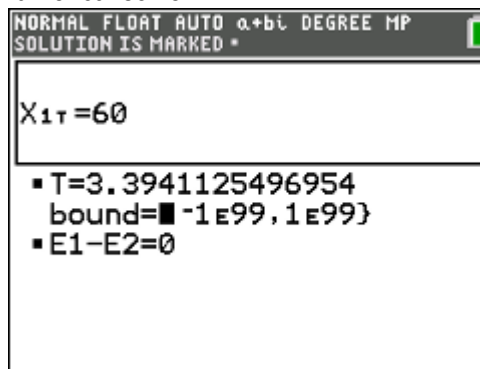
which means $t \approx 3.3941125496954\dots$

The value for t should be stored and used in the calculations as shown opposite. As the multiple choice responses provide options for 4.5 and 4.6 the latter should be chosen as rounding 4.552 produces 4.6 as the preferred answer.

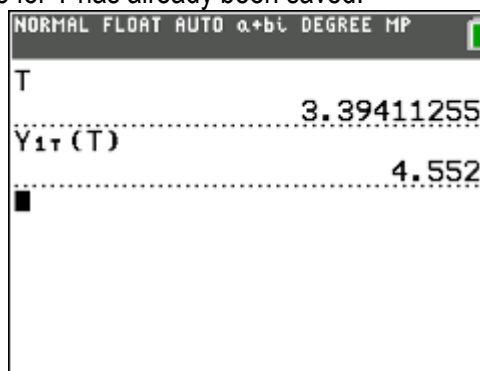
It is rare for examinations to focus on this level of accuracy, however it is good practice to retain as much accuracy as possible during calculations by storing results.

The calculate option in the Parametric Graph mode is available for T only.

Use the numerical solver:



This value can now be substituted into Y_1 . Remember that the value for T has already been saved.



Question 12. Answer D

$$\mathbf{r}(t) = 10 \cos(t)\mathbf{i} + 10\sin(t)\mathbf{j} + t^2\mathbf{k}$$

$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = -10 \sin(t)\mathbf{i} + 10 \cos(t)\mathbf{j} + 2t\mathbf{k}$$

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = -10 \cos(t)\mathbf{i} - 10 \sin(t)\mathbf{j} + 2\mathbf{k}$$

$$\text{Speed} = |\mathbf{v}(t)|$$

$$= \sqrt{(-10 \sin(t))^2 + (10 \cos(t))^2 + (2t)^2}$$

$$= \sqrt{100 (\sin(t))^2 + 100(\cos(t))^2 + 4t^2}$$

$$= \sqrt{100(\sin(t))^2 + (\cos(t))^2 + 4t^2}$$

$$= \sqrt{100 + 4t^2}$$

The speed is always increasing over the interval, so the maximum speed occurs at $t = 6\pi$ seconds.

$$\text{Speed} = \sqrt{100 + 4 \times (6\pi)^2}$$

$$= 39\text{m/sec}$$

Velocity and acceleration at right angles when the dot product = 0.

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = 0$$

$$-10 \sin(t) \times -10 \cos(t) + 10 \cos(t) \times -10 \sin(t) + 2t \times 2 = 0$$

$$100 \sin(t) \times \cos(t) - 100\sin(t) \times \cos(t) + 4t = 0$$

$$4t = 0$$

$$t = 0 \text{ seconds}$$

The velocity and acceleration are at rightangles at the start. ($t = 0$)

There are no conventional ways this can be done on the TI-84PlusCE as the vectors are three dimensional which in turn eliminates the parametric equations.

Students need to do the calculus components by hand leaving on the expression:

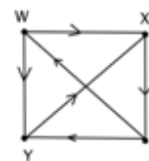
$\sqrt{100 + 4t^2}$ which can be resolved on inspection as an increasing function, therefore the maximum occurs when $t = 6\pi$

Matrices

Simple Familiar

Question 1.

Use a one-step and two-step dominance matrix ($D + \frac{1}{4}D^2$) to rank the four players (W, X, Y and Z) who competed in a round-robin competition represented by the digraph.

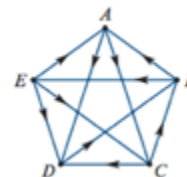


The ranking from best to worst is:

- a) ZWXY b) ZWYX c) WZXY d) WZYX e) WYXZ

Question 2.

Five teams competed in one round of a Futsal competition, with each team playing every other team once. The digraph represents the results of the competition.



Use the weighting $D + \frac{1}{2}D^2 + \frac{1}{10}D^3$ applied to the dominance scores to determine the overall ranking of the teams.

Which one of the following represents the ranking of the teams from first to last?

- a) BDAEC b) BEDAC c) EBDAC d) EBDCA e) EBCAD

Question 3.

Researchers found that a colony of bats live for only 3 years. Birth rate and death rate information is contained in the table below.

Age Group	0-1 year	1-2 years	2-3 years
Birth Rate	2	4	0
Death Rate	0.4	0.3	1

Which one of the following is the Leslie matrix which represents the information contained in the table?

- A. $\begin{bmatrix} 2 & 4 & 0 \\ 0.6 & 0 & 0 \\ 0.7 & 0 & 0 \end{bmatrix}$ B. $\begin{bmatrix} 2 & 4 & 0 \\ 0.4 & 0 & 0 \\ 0.3 & 0 & 0 \end{bmatrix}$ C. $\begin{bmatrix} 2 & 4 & 0 \\ 0.6 & 0.7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ D. $\begin{bmatrix} 2 & 4 & 0 \\ 0.6 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}$ E. $\begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 0 \\ 0.6 & 0.7 & 0 \end{bmatrix}$

Question 4.

Given the Leslie Matrix L and the initial population P_0 :

$$L = \begin{bmatrix} 10 & 8 & 6 \\ 0.8 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \text{ and } P_0 = \begin{bmatrix} 100 \\ 80 \\ 10 \end{bmatrix}$$

Which one of the following represents $L^2 P_0$?

A $\begin{bmatrix} 1700 \\ 80 \\ 24 \end{bmatrix}$ B $\begin{bmatrix} 17784 \\ 1360 \\ 24 \end{bmatrix}$ C $\begin{bmatrix} 1700 \\ 1360 \\ 24 \end{bmatrix}$ D $\begin{bmatrix} 17784 \\ 1630 \\ 24 \end{bmatrix}$ E $\begin{bmatrix} 1700 \\ 80 \\ 0 \end{bmatrix}$

Question 5.

The Leslie Matrix L and the initial population P_0 represent a female rat population which have a lifespan of 3 years.

$$L = \begin{bmatrix} 1.6 & 4.2 & 1.2 \\ 0.6 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \text{ and } P_0 = \begin{bmatrix} 20 \\ 12 \\ 6 \end{bmatrix}$$

The rat population is divided into 3 age groups- young, juveniles and adults with each age group representing 1 year. How many years will it take the female rat population to pass 1,000,000 in number?

- a) 11 b) 12 c) 33 d) 36 e) 39

Question 6.

The birth rate and death rate for a certain species of grasshopper is given in the table below.

Age (months)	0 - 3	3 - 6	6 - 9	9 - 12
Number	18	26	15	8
Birth rate	0	1.5	2.3	0
Death rate	0.3	0.2	0.4	1

Which one of the following represents the long-term behaviour of the population? In other words, what percent does the population growth rate approach?

- a) 125% b) 130% c) 135% d) 140% e) 145%

Question 7.

Determine the unique solution for the system of equations below by using an augmented matrix and the Reduced Row Echelon command $rref()$.

$$\begin{aligned} x - y + 2z &= 7 \\ -x - 2y + 3z &= 4 \\ -2x - 2y + z &= -6 \end{aligned}$$

- a) $(x, y, z) = (2, 3, 4)$ b) $(x, y, z) = (-2, -3, -4)$
c) $(x, y, z) = (2, -3, -4)$ d) $(x, y, z) = (-2, -3, 4)$
e) $(x, y, z) = (-2, 3, -4)$

Complex Familiar

Question 8.

A kangaroo population is described by the data below.

Age (months)	0 - 3	3 - 6	6 - 9	9 - 12
Number	18	26	15	8
Birth rate	0	1.5	2.3	0
Death rate	0.3	0.2	0.4	1

For this changing kangaroo population, the growth factor is defined as the ratio of the total female population at a given period P_{N+1} to that of the previous period P_N for large N , that is:

$$R = \frac{P_{N+1}}{P_N}, N \geq 0. \text{ For an increasing population } R > 1.$$

To maintain a controlled, stable population ($R = 1$), a harvesting or culling factor h is applied such that:

$$h = 1 - \frac{1}{R}$$

Which one of the following represents a suitable culling factor?

- a) 125% b) 130% c) 135% d) 140% e) 145%

Question 9.

Use row operations to solve the following system of equations:

$$\begin{aligned}x + z &= 2y - 2 \\3x + 5y &= 20 - 2z \\5x + 2y - 3z &= 27\end{aligned}$$

- a) $(x, y, z) = \left(\frac{15}{4}, \frac{9}{4}, \frac{5}{4}\right)$ b) $(x, y, z) = \left(\frac{15}{4}, \frac{-9}{4}, \frac{-5}{4}\right)$
c) $(x, y, z) = \left(\frac{15}{4}, \frac{9}{4}, \frac{-5}{4}\right)$ d) $(x, y, z) = \left(\frac{-15}{4}, \frac{9}{4}, \frac{-5}{4}\right)$
e) $(x, y, z) = \left(\frac{-15}{4}, \frac{-9}{4}, \frac{-5}{4}\right)$

Answers

1. A	2. E	3. D	4. B	5. C	6. D	7. A	8. E	9. A
------	------	------	------	------	------	------	------	------

Question 1. Answer A

The matrix **A** represents a one-step dominance.

Matrix **b** is used to help collate the total of the one-step dominance and two-step dominance matrices.

$(A + 0.25 A^2) \times B$ will tally the results for ranking.

The result shows the rankings are:
Z, W, X, and Y.

NORMAL FLOAT AUTO a+bi DEGREE MP

MATRIX[A] 4 x4

0	1	1	0
0	0	0	1
0	1	0	0
1	0	1	0

NORMAL FLOAT AUTO a+bi DEGREE MP

MATRIX[B] 4 x1

1
1
1
1

NORMAL FLOAT AUTO a+bi DEGREE MP

$([A] + .25[A]^2) * [B]$

2.5
1.5
1.25
2.75

Question 2. Answer E

First determine and define the one-step dominance matrix **A**.

Apply the weighting and multiply by the column vector **B** to summarise the information.

Ranking is E, B, C, A, D

NORMAL FLOAT AUTO a+bi DEGREE MP

MATRIX[A] 5 x5

0	0	1	1	0
1	0	0	0	1
0	1	0	1	0
0	1	0	0	0
1	0	1	1	0

NORMAL FLOAT AUTO a+bi DEGREE MP

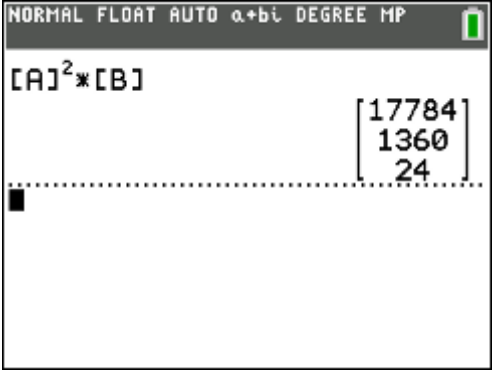
$([A] + .5[A]^2 + .1[A]^3) * [B]$

1.25
2.75
4
5.3
4.2
2.5
6.3

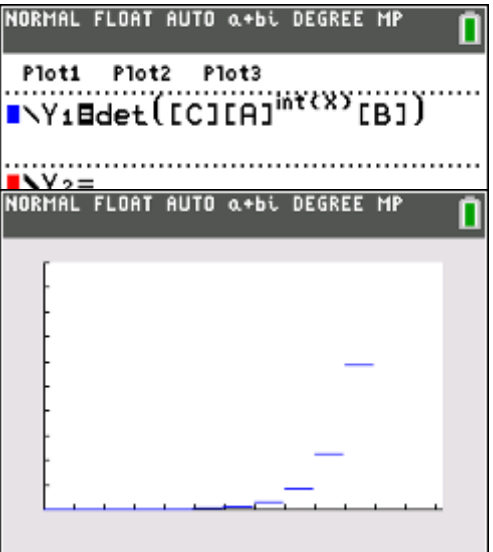
Question 3. Answer D

This question does not require calculator functionality, it has however been included here as a precursor to subsequent questions that assume you can generate the Leslie Matrix from the given information.

Question 4. Answer B

$l := \begin{bmatrix} 10 & 8 & 6 \\ 0.8 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}$ $p := \begin{bmatrix} 100 \\ 80 \\ 10 \end{bmatrix}$ $l^2 \cdot p \quad \begin{bmatrix} 17784. \\ 1360. \\ 24. \end{bmatrix}$	<p>Matrices defined as shown on the left with A = L and B = P.</p> 
---	---

Question 5. Answer A

<p>First define the Leslie matrix (L) and initial population matrix P₀.</p> <p>Try some values in the multi-choice answers, for example when n = 9 (27 years) and n = 12 (36 years).</p> <p>Use a row matrix to determine the total sum.</p> $l := \begin{bmatrix} 1.6 & 4.2 & 1.2 \\ 0.6 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$ $p := \begin{bmatrix} 20 \\ 12 \\ 6 \end{bmatrix}$ <table border="0" style="width: 100%;"> <tr> <td>$e \cdot l^{12} \cdot p$</td> <td>$[3.99813E6]$</td> </tr> <tr> <td>$e \cdot l^{11} \cdot p$</td> <td>$[1.52838E6]$</td> </tr> <tr> <td>$e \cdot l^9 \cdot p$</td> <td>$[223347.]$</td> </tr> <tr> <td>$e := [1 \ 1 \ 1]$</td> <td>$[1 \ 1 \ 1]$</td> </tr> <tr> <td>$e \cdot l^{10} \cdot p$</td> <td>$[584254.]$</td> </tr> </table>	$e \cdot l^{12} \cdot p$	$[3.99813E6]$	$e \cdot l^{11} \cdot p$	$[1.52838E6]$	$e \cdot l^9 \cdot p$	$[223347.]$	$e := [1 \ 1 \ 1]$	$[1 \ 1 \ 1]$	$e \cdot l^{10} \cdot p$	$[584254.]$	<p>The same calculator approach shown on the TI-Nspire can be used on the TI-84PlusCE.</p> <p>An alternative, visual approach can be used in the Y= editor. The bulk of the expression is the same, with two main differences:</p> <p>DET = Determinant (Numerical quantity) INT = Integer (Only integer powers permitted)</p>  <p>Note: The determinant of a 1 x 1 matrix is a neat way of simply extracting the numerical quantity. Using integer (INT) makes it easier to see the graph. Use Trace to locate the first time the population exceeds 1,000,000</p>
$e \cdot l^{12} \cdot p$	$[3.99813E6]$										
$e \cdot l^{11} \cdot p$	$[1.52838E6]$										
$e \cdot l^9 \cdot p$	$[223347.]$										
$e := [1 \ 1 \ 1]$	$[1 \ 1 \ 1]$										
$e \cdot l^{10} \cdot p$	$[584254.]$										

Question 6. Answer D

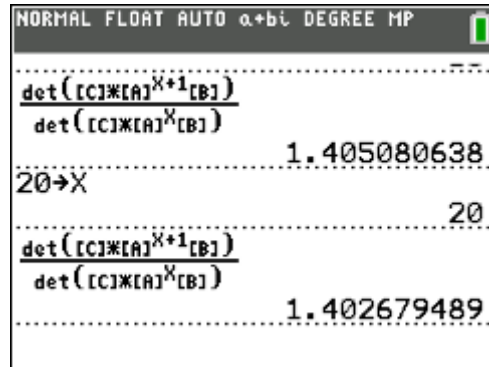
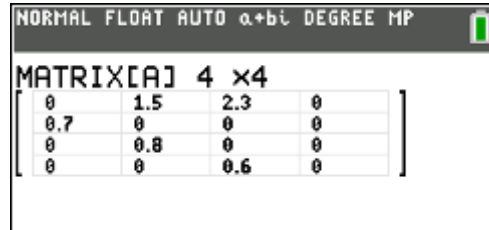
First determine and define the Leslie matrix (L) and the initial population matrix (P_0)
 Also define matrix e so the population in each interval can be summed to get the total population.

Determine the ratio of a long-term distributions, say 20 and 21 cycles.

$$\text{Eg } \frac{L^{21}P_0}{L^{20}P_0} \times 100 = 140.3\%$$

Therefore % increase is 40.3%

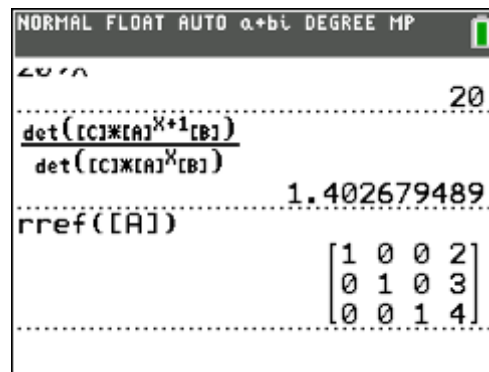
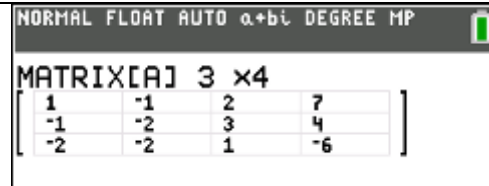
With matrices defined in A, B and C, the long term behaviour can easily be observed by progressively increasing the value of x in the ratio (below).



Question 7. Answer A

From using the **rref()** command:

$$x = 2, y = 3 \text{ and } z = 4$$



Question 8. Answer E

First define the matrices L, P₀ and E.

$$L := \begin{bmatrix} 0 & 0 & 3.9 & 2.7 & 0.9 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix} \quad p_0 := \begin{bmatrix} 3400 \\ 2500 \\ 2300 \\ 1750 \\ 650 \end{bmatrix} \quad e := [1 \ 1 \ 1 \ 1 \ 1]$$

Determine long term growth rate.

$$\text{Eg } \frac{L^{51}P_0}{L^{50}P_0} \times 100 = 130.05$$

$$\text{Eg } \frac{L^{51}P_0}{L^{50}P_0} \times 100 = 130.07$$

Now $h = 1 - \frac{1}{R}$ and $R \approx 1.301$

$$h = 1 - \frac{1}{1.301}$$

$$h = 0.231$$

Culling factor should be 23.1%

The calculator screen shows the following sequence of operations and results:

- Mode: NORMAL FLOAT AUTO a+bi DEGREE MP
- Result: 1.308548419
- Input: 50 → X
- Result: 50
- Expression: $\frac{\det([C] * [A]^{X+1} [B])}{\det([C] * [A]^X [B])}$
- Result: 1.30051078
- Input: $1 - \frac{1}{\text{Ans}}$
- Result: 0.2310713489

Note the use of [Ans] to access the result of the previous calculation, the previous calculation can also be pasted or accessed by using [RCL] + [Ans]. Performing calculations using this technique maintains maximum accuracy.

Tip: The numerator from the expression for the long term relationship can be used in the Y= editor. By using the numerator only, calculating $Y_1(50)/Y_1(49)$ would determine the long term ratio. As an equation in the Y= editor, it can be stored as a very useful Graphical Data Base which could consist of multiple, similar expressions!

Question 9. Answer C

First rearrange the equations so the variables are all on the left-hand side and the constants are on the right.

$$\begin{aligned}x - 2y + z &= -2 \\3x + 5y + 2z &= 20 \\5x + 2y - 3z &= 27\end{aligned}$$

Define an augmented matrix M

Use row operations

$$-5 \times R_1 + R_3 \rightarrow R_3$$

$$-3 \times R_1 + R_2 \rightarrow R_2$$

$$-\frac{12}{11} \times R_2 + R_3 \rightarrow R_3$$

$$\frac{-76}{11} \times R_3 \rightarrow R_3$$

$$R_3 + R_2 \rightarrow R_2$$

$$\frac{1}{11} \times R_2 \rightarrow R_2$$

NORMAL FLOAT AUTO a+bi DEGREE MP

MATRIX[A] 3 × 4

$$\begin{bmatrix} 1 & -2 & 1 & -2 \\ 3 & 5 & 2 & 20 \\ 5 & 2 & -3 & 27 \end{bmatrix}$$

NORMAL FLOAT AUTO a+bi DEGREE MP

0.2310/13989

*row+(-5,[A],1,3)→[A]

$$\begin{bmatrix} 1 & -2 & 1 & -2 \\ 3 & 5 & 2 & 20 \\ 0 & 12 & -8 & 37 \end{bmatrix}$$

*row+(-3,[A],1,2)→[A]

$$\begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 11 & -1 & 26 \\ 0 & 12 & -8 & 37 \end{bmatrix}$$

NORMAL FLOAT AUTO a+bi DEGREE MP

$$\begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 11 & -1 & 26 \\ 0 & 12 & -8 & 37 \end{bmatrix}$$

*row+(-12/11,[A],2,3)→[A]

$$\begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 11 & -1 & 26 \\ 0 & 0 & -76/11 & 95/11 \end{bmatrix}$$

NORMAL FLOAT AUTO a+bi DEGREE MP

$$\begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 11 & -1 & 26 \\ 0 & 0 & -76/11 & 95/11 \end{bmatrix}$$

*row(-11/76,[A],3)

$$\begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 11 & -1 & 26 \\ 0 & 0 & 1 & -5/4 \end{bmatrix}$$

...

This process can continue until the final matrix consists of a leading diagonal of ones. It is important to note however that for a multiple choice question it is only the final answer that is viewed, so RREF could be used in preference in this case.