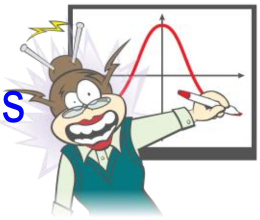
Exam Preparation – Extended Response Questions



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Each of the questions included here can be solved using TI-Nspire CX CAS technology.

Poll Question

Consider the differential equation $\frac{dy}{dx} = x^2 + \sqrt{y}$, where $y(0) = 2$. Using Euler's method with a step size of 0.05, the value of $y(0.25) = y_5$ is best approximated by

A. 0.0092
B. 1.6628
C. 2.3697
D. 2.2941
E. 1.7284

Question 1

Let $f : D \to \mathbb{R}$, $f(x) = \frac{25x^2}{x^3 + 64}$, where D is the implied domain of f.

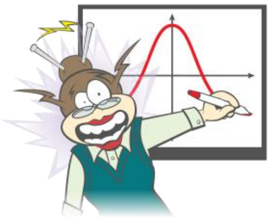
a. i. Find D.

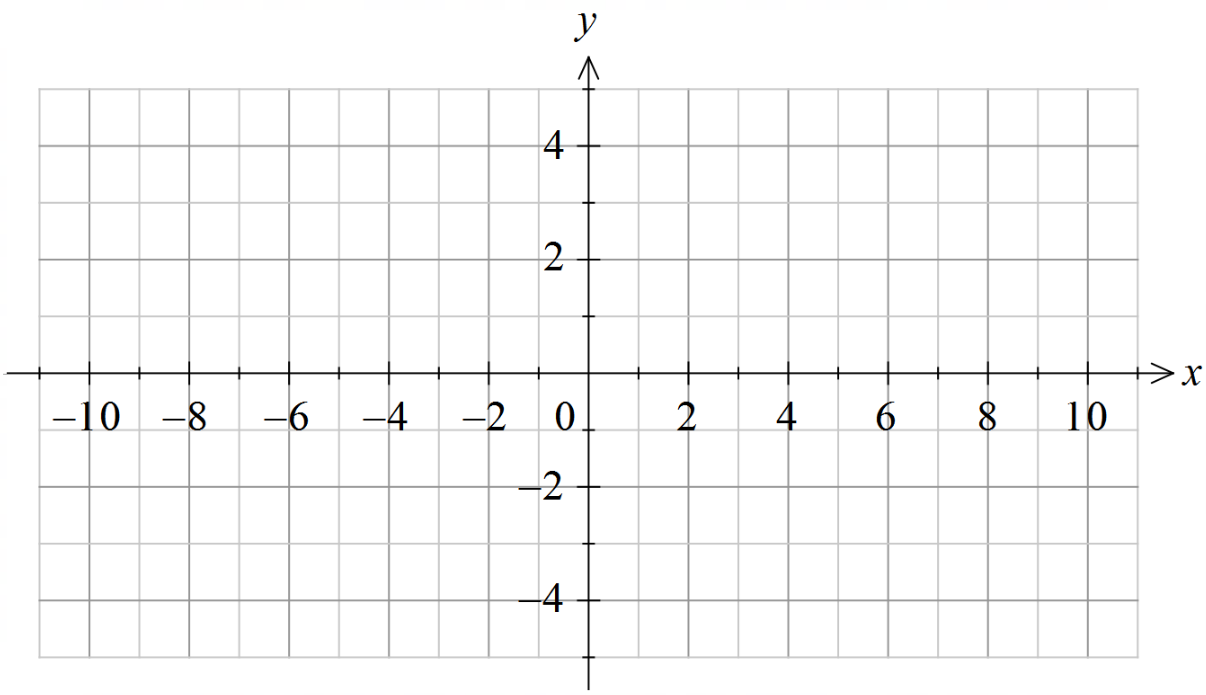
ii. Find the coordinates of the stationary points on the graph of $y = f(x)$, correct to two decimal places.

iii. Find the coordinates of the points of inflection on the graph of $y = f(x)$, correct to two decimal places.

Response:

b. Sketch the graph of $y = f(x)$ on the axes below, for $x \in [-10, 10]$, labelling the stationary points and points of inflection correct to two decimal places. Label the asymptotes with their equations and axes intercepts with their exact coordinates.





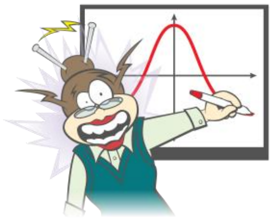
c. A beverage glass is to be modelled by rotating the region bounded by the graph of $y = f(x)$, the x-axis and the line $x = 9$ about the x-axis. All measurements are in centimetres.

i. Write down a definite integral, which when evaluated, will give the volume of the glass in cm^3.

ii. Find the volume of the glass, correct to two decimal places.

iii. The glass is positioned upright and is filled with beverage such that it fills exactly three-fifths of the glass. Determine the depth, d cm, of beverage in the glass, correct to three decimal places.

Response:



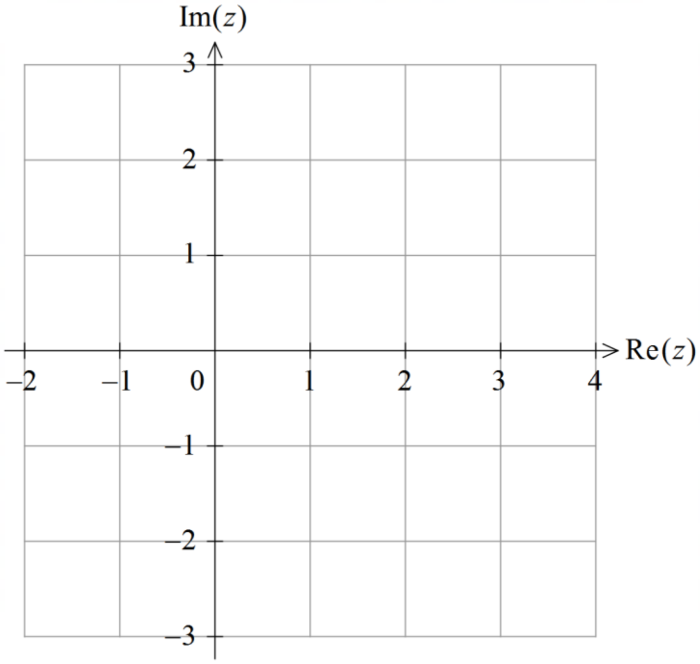
Question 2

A straight line in the complex plane is given by $|z-1|=|z+1-\sqrt{2}i|,\ z\in\mathbb{C}$.

a. Find the Cartesian equation of this line, in the form $y=\dfrac{\sqrt{a}}{a}(ax+b)$, where $a,b\in\mathbb{Z}^+$.

b. Find the coordinates of the points of intersection of the line $|z-1|=|z+1-\sqrt{2}i|$ with the circle $|z-1|=2$. Give answers correct to three decimal places.

c. On the Argand diagram below, sketch the graphs of the line $|z-1|=|z+1-\sqrt{2}i|$ and the circle $|z-1|=2$, showing all coordinates of points of intersection correct to three decimal places.



d. A ray in the complex plane is defined by $\text{Arg}(z)=\alpha$, where $-\pi<\alpha\leq\pi$.

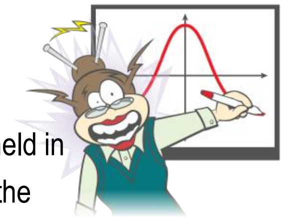
i. When $\alpha=\dfrac{\pi}{4}$, the ray intersects the above circle once. Find the exact coordinates of the point of intersection.

ii. State the range of values of α for which the ray intersects both the above circle and the above line.

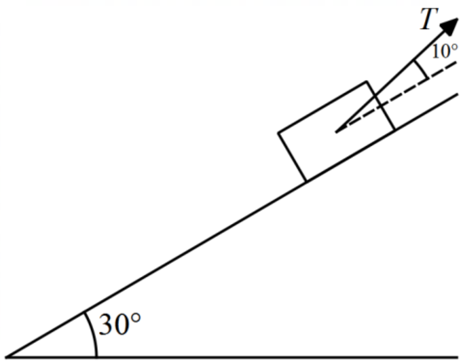
e. The region S exists in the first quadrant, and is bounded by the y-axis, the line $|z-1|=|z+1-\sqrt{2}i|$ and the circle $|z-1|=2$. Find the area of S, correct to two decimal places.

Response:

Question 3



A block of mass 20 kg is at rest on a smooth plane inclined at 30° to the horizontal. It is held in place by a light, inextensible string that exerts a force of T newtons at an angle of 10° to the plane, as shown in the diagram below.



a. Draw and label all the other forces acting on the block on the diagram above.
b. Find the value of T, correct to one decimal place.

The block is situated 40 m from the bottom of the plane. The rope is cut so that the block slides from rest down the plane. It is resisted by a force of v^2 newtons acting up the plane, where $v\text{ m/s}$ is the speed of the block.

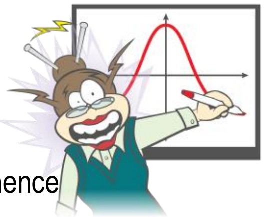
c. By resolving forces parallel to the plane, show that the acceleration of the block down the plane is $\dfrac{98-v^2}{20}\text{ m/s}^2$.

d. Show that a differential equation that relates the speed $v\text{ m/s}$ of the block to the distance $x\text{ m}$ that it has travelled is $\dfrac{dv}{dx}=\dfrac{98-v^2}{20v}$, and hence find v in terms of x, in the form $v=\sqrt{p-pe^{qx}}$, where $p,q\in\mathbb{Q}$.

e. It takes T seconds for the block to reach the bottom of the inclined plane. Write a definite integral for T, and hence evaluate this quantity, correct to the nearest integer.

Response:

Question 4



A large vat initially contains $1000\ \mathrm{L}$ of pure water. Syrup, with a concentration of $0.1\ \mathrm{kg}$ of sugar per litre, enters the tank at a rate of $8\ \mathrm{L/s}$. The mixture is stirred continuously and is hence kept uniform, and leaves the vat at a rate of $4\ \mathrm{L/s}$.

a. At time t seconds, the mass of sugar in the vat is x kilograms.

i. Write an expression for the concentration of sugar in the vat in terms of x and t.

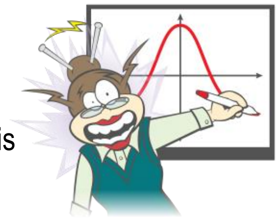
ii. Show that the differential equation relating x and t is $\frac{dx}{dt} = \frac{4}{5} - \frac{x}{t+250}$.

b. It can be shown that $x = \frac{2t(t+500)}{5(t+250)}$. Verify, by substitution, that the given solution satisfies both the differential equation and the initial condition.

c. Find the time it takes, in seconds and correct to one decimal place, for the mass of sugar to reach $100\ \mathrm{kg}$.

d. By using a definite integral, find the total amount of sugar that flows out of the vat in the first 100 seconds.

Response:



Question 5

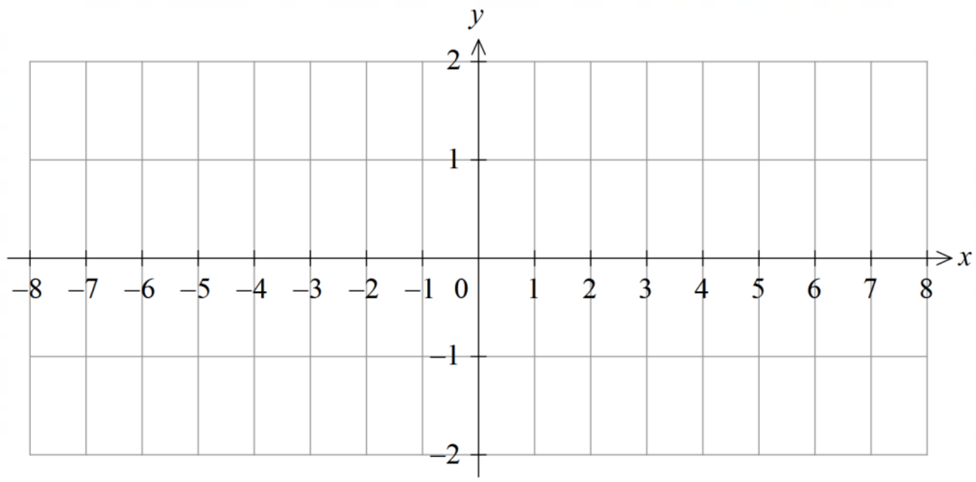
Relative to the origin O, the position vector of a model tram on a circuit at time t seconds is

$$\underset{\sim}{r}(t) = 6\sin\left(\frac{\pi t}{8}\right)\underset{\sim}{i} + \left(\sin\left(\frac{\pi t}{4}\right) - \cos\left(\frac{\pi t}{8}\right)\right)\underset{\sim}{j}$$

where $t \in [0, \infty)$ and measurements are in metres. $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the direction of the positive x- and y-axes respectively.

a. By taking the square of the $\underset{\sim}{j}$ component and using appropriate double-angle formulas, find the Cartesian equation of the circuit, in the form $cy^2 = (a - x^2)(x - b)^2$, where $a, b, c \in \mathbb{Z}^+$.

b. Sketch the path of the model tram on the axes below. Label the initial position of the tram with its coordinates, and the initial direction of motion of the tram with an arrow.



c. Let $\underset{\sim}{v}(t) = \dot{\underset{\sim}{r}}(t)$ be the velocity of the tram at time t.

i. Find the first time when the speed of the tram is at a minimum, and find this minimum speed correct to two decimal places.

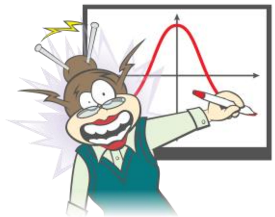
ii. State the coordinates of the tram's position on the circuit at this time.

d. Let L metres be the total length of one 'lap' of the circuit.

i. Write a definite integral which, when evaluated, gives the value of L.

ii. Hence, find L, correct to three decimal places.

Response:



Answers*

Question 4

a.i. The volume of mixture in the vat at time t is $1000+8t-4t=1000+4t$, so the concentration is $\dfrac{\text{mass}}{\text{volume}}=\dfrac{x}{1000+4t}$.

a.ii. The rate of inflow of sugar is $0.1\text{ kg/L}\times 8\text{ L/s}=\dfrac{4}{5}\text{ kg/s}$.

The rate of outflow of sugar is

$$\frac{x}{1000+4t}\text{ kg/L}\times 4\text{ L/s}=\frac{x}{t+250}\text{ kg/s}.$$

Therefore the overall rate of change of sugar is given by the differential equation $\dfrac{dx}{dt}=\dfrac{4}{5}-\dfrac{x}{t+250}$, as required.

b. For $x=\dfrac{2t(t+500)}{5(t+250)}=\dfrac{2t^2+1000t}{5t+1250}$, the derivative is, by the quotient rule,

$$\frac{dx}{dt}=\frac{(4t+1000)(5t+1250)-(5)(2t^2+1000t)}{(5t+1250)^2}$$

$$=\frac{2(t^2+500t+125000)}{5(t+250)^2}$$

$$\frac{4}{5}-\frac{x}{t+250}=\frac{4}{5}-\frac{1}{t+250}\times\frac{2t(t+500)}{5(t+250)}$$

$$=\frac{4}{5}-\frac{2t(t+500)}{5(t+250)^2}$$

$$=\frac{4(t+250)^2-2t(t+500)}{5(t+250)^2}$$

$$=\frac{2(t^2+500t+125000)}{5(t+250)^2}$$

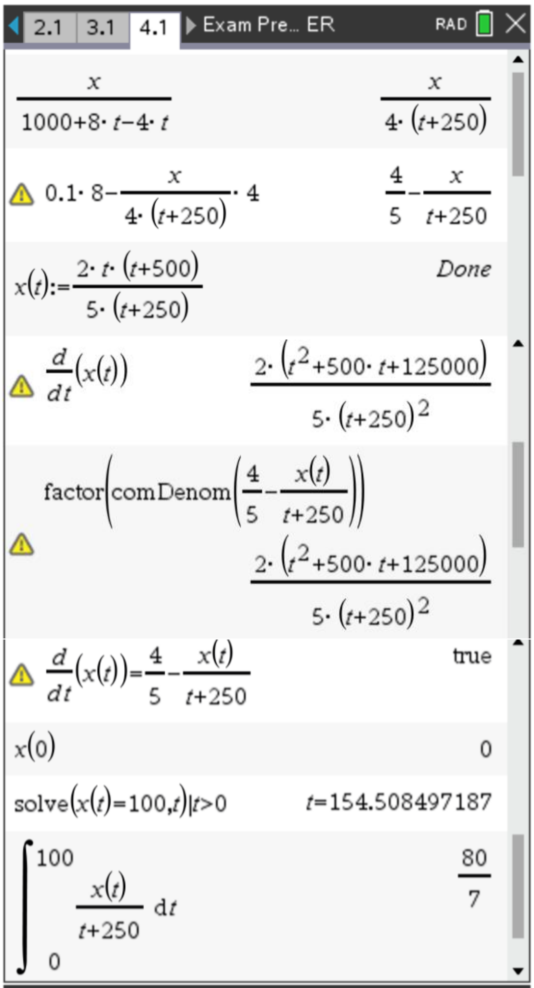
$$x(0)=\frac{2(0)(0+500)}{5(0+250)}=0$$

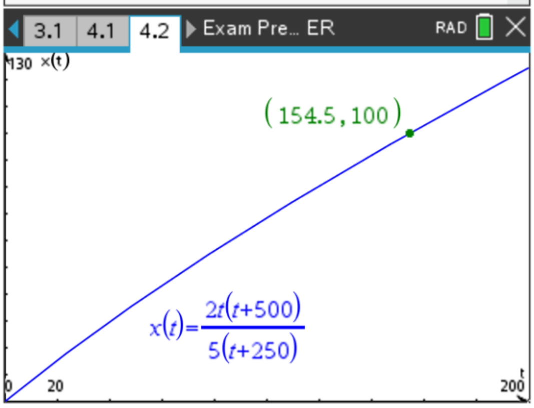
Therefore both the DE and the IC are satisfied.

c. Solving $x(t)=100$ for $t>0$ gives $t\approx 154.5$ s (1 d.p.).

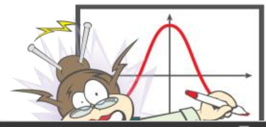
d. The rate of outflow of sugar at time t is given by $\dfrac{x}{t+250}$, so the amount of sugar flowing out in the first 100 s is

$$\int_0^{100}\frac{x}{t+250}\,dt=\int_0^{100}\frac{2t(t+500)}{5(t+250)^2}\,dt=\frac{80}{7}\text{ kg}.$$





* When using CAS as a calculation and/or algebraic manipulation tool, it is important to set out working clearly.



Question 5

a. $x = 6\sin\left(\frac{\pi t}{8}\right) \Rightarrow \frac{x}{6} = \sin\left(\frac{\pi t}{8}\right)$

$y = \sin\left(\frac{\pi t}{4}\right) - \cos\left(\frac{\pi t}{8}\right)$

$$y^2 = \left(\sin\left(\frac{\pi t}{4}\right) - \cos\left(\frac{\pi t}{8}\right)\right)^2$$

$$= \sin^2\left(\frac{\pi t}{4}\right) - 2\sin\left(\frac{\pi t}{4}\right)\cos\left(\frac{\pi t}{8}\right) + \cos^2\left(\frac{\pi t}{8}\right)$$

$$= \left(2\sin\left(\frac{\pi t}{8}\right)\cos\left(\frac{\pi t}{8}\right)\right)^2$$

$$-2\left(2\sin\left(\frac{\pi t}{8}\right)\cos\left(\frac{\pi t}{8}\right)\right)\cos\left(\frac{\pi t}{8}\right) + \cos^2\left(\frac{\pi t}{8}\right)$$

$$= 4\sin^2\left(\frac{\pi t}{8}\right)\cos^2\left(\frac{\pi t}{8}\right)$$

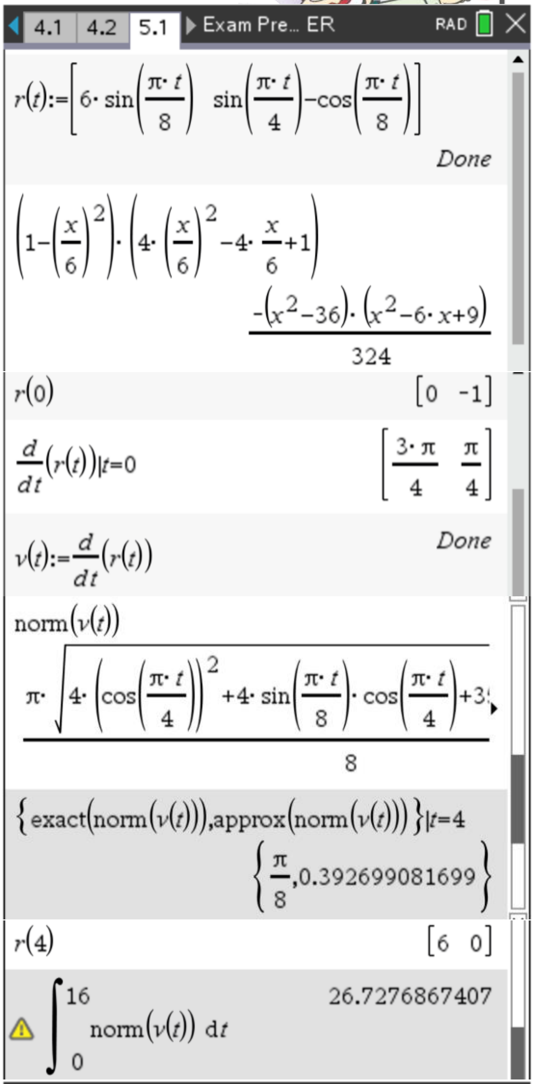
$$-4\sin\left(\frac{\pi t}{8}\right)\cos^2\left(\frac{\pi t}{8}\right) + \cos^2\left(\frac{\pi t}{8}\right)$$

$$= \cos^2\left(\frac{\pi t}{8}\right)\left(4\sin^2\left(\frac{\pi t}{8}\right) - 4\sin\left(\frac{\pi t}{8}\right) + 1\right)$$

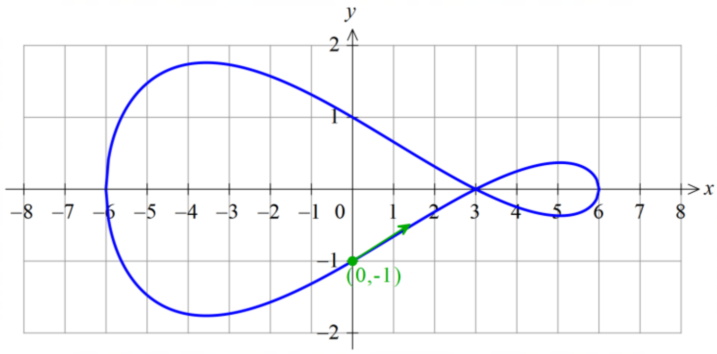
$$= \left(1 - \sin^2\left(\frac{\pi t}{8}\right)\right)\left(4\sin^2\left(\frac{\pi t}{8}\right) - 4\sin\left(\frac{\pi t}{8}\right) + 1\right)$$

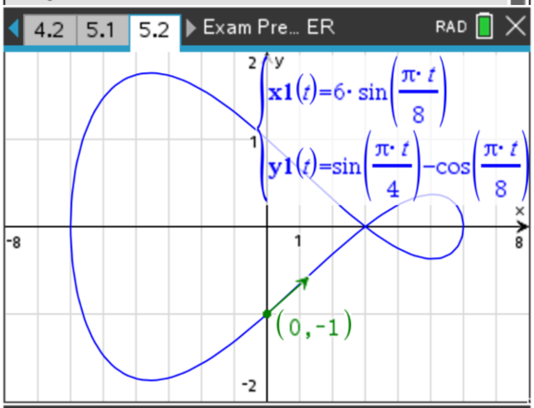
$$= \left(1 - \left(\frac{x}{6}\right)^2\right)\left(4\left(\frac{x}{6}\right)^2 - 4\frac{x}{6} + 1\right)$$

$$324y^2 = \left(36 - x^2\right)\left(x - 3\right)^2$$



b.





c.i. Analysing the graph of $\left|\underset{\sim}{v}(t)\right|$, the tram's speed is first a minimum when $t = 4$. At this time, the speed is 0.39 m/s.

c.ii. $\underset{\sim}{r}(4) = 6\underset{\sim}{i}$ so the tram is at $(6, 0)$.

d.i. The arc length of a lap of the circuit is $\int_0^{16}\sqrt{\left|\dot{\underset{\sim}{r}}(t)\right|}dt$ where $\left|\dot{\underset{\sim}{r}}(t)\right| = \left|\underset{\sim}{v}(t)\right|$ is the speed of the tram and 16 is the time taken for one lap (period). Therefore $L = \int_0^{16}\sqrt{\left|\underset{\sim}{v}(t)\right|}dt$.

d.ii. $L = 26.728$ m (3 d.p.).