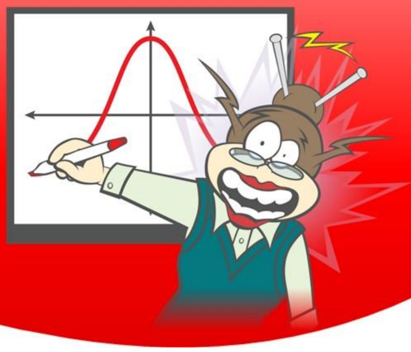

Vectors Part 2

Each of the questions included here can be solved using the TI-Nspire CX CAS.

Scan the QR code or use the link: http://bit.ly/Vectors-Part2

Question: 1.

Evaluate the scalar product of $\underset{\sim}{a} = 5\underset{\sim}{i} - 3\underset{\sim}{k}$ in the direction of $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$.

Question: 2.

Express the vector $\underset{\sim}{v} = 3\underset{\sim}{i} + 6\underset{\sim}{j} - 2\underset{\sim}{k}$ as a sum of two vectors, one of which is parallel to the vector $\underset{\sim}{w} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and one which is perpendicular to it.

Question: 3.

The vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $5\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ and the vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\mu\underset{\sim}{i} + 2\underset{\sim}{j} - 9\underset{\sim}{k}$. Show that the value of μ is 3 and hence determine the vector $\underset{\sim}{a}$.

Author: S Crouch

Question: 4.

Determine if the vectors $\underset{\sim}{a}=3\underset{\sim}{i}-\underset{\sim}{j}+5\underset{\sim}{k}$, $\underset{\sim}{b}=2\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}$ and $\underset{\sim}{c}=4\underset{\sim}{i}+3\underset{\sim}{k}-6\underset{\sim}{k}$ are linearly independent.

Question: 5.

If the vectors $\underset{\sim}{p}=\underset{\sim}{i}-2\underset{\sim}{j}$, $\underset{\sim}{q}=4\underset{\sim}{i}+8\underset{\sim}{k}$ and $\underset{\sim}{r}=\alpha\underset{\sim}{i}-\underset{\sim}{j}+5\underset{\sim}{k}$ form a linearly dependent set of vectors, find the exact value of α.

Question: 6.

The vectors $\underset{\sim}{a}=2\underset{\sim}{i}-3\underset{\sim}{j}+\underset{\sim}{k}$, $\underset{\sim}{b}=5\underset{\sim}{i}+4\underset{\sim}{j}-6\underset{\sim}{k}$ and $\underset{\sim}{c}=4\underset{\sim}{i}-29\underset{\sim}{j}+19\underset{\sim}{k}$ are linearly dependent. Write $\underset{\sim}{c}$ as a linear combination of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

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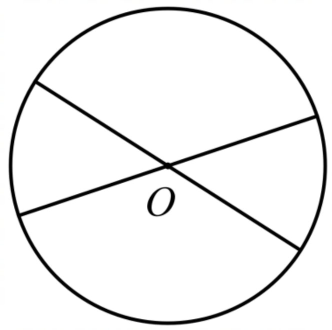
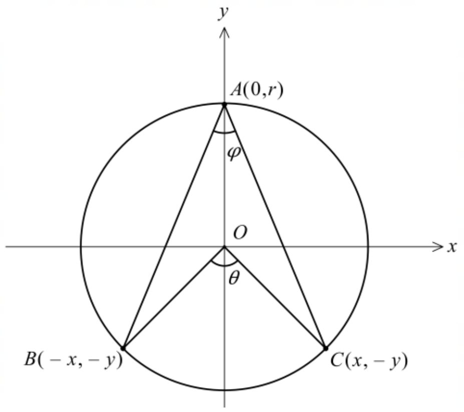
Question: 7.

Use vectors to prove that the quadrilateral formed by the endpoints of two non-concurrent diameters of a circle is a rectangle.



Question: 8.

The diagram below shows a circle of radius r, centred at $O(0,0)$ on the Cartesian plane. The points $A(0,r)$, $B(-x,-y)$ and $C(x,-y)$ all lie on the circle, where $r, x, y \in \mathbb{R}^+$. Let φ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and let θ be the angle between $\overrightarrow{OB}$ and $\overrightarrow{OC}$.



Show that $\cos(\theta) = \dfrac{y^2 - x^2}{x^2 + y^2}$ and $\cos(\varphi) = \dfrac{y}{\sqrt{x^2 + y^2}}$. Hence, prove that $\theta = 2\varphi$.

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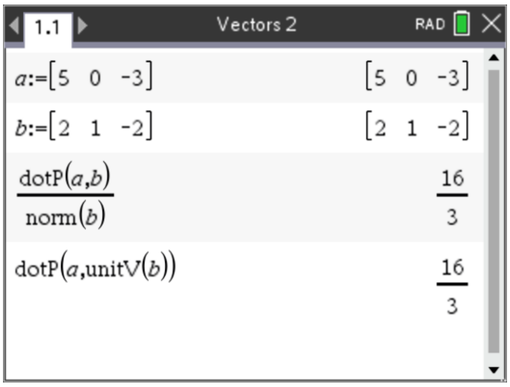
Answers

Question 1 $\frac{16}{3}$

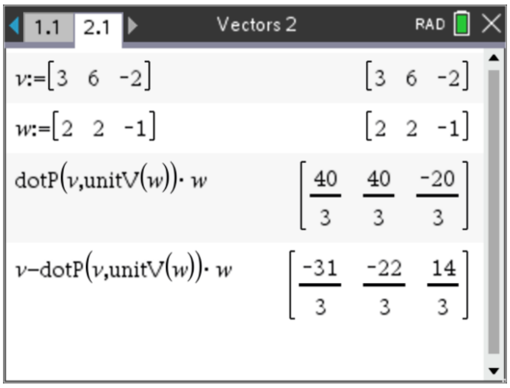


Define the vectors. [ctrl] [:=] can be used to get :=

The scalar resolute of $\underset{\sim}{a}$ parallel to $\underset{\sim}{b}$ is $\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|}=\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}$.

The Dot Product can be found using the dotP command ([menu] [7] [C] [3]). Magnitudes of vectors can be found using the norm command ([menu] [7] [7] [1]).

Question 2 $\underset{\sim}{v}=\left(\frac{40}{3}\underset{\sim}{i}+\frac{40}{3}\underset{\sim}{j}-\frac{20}{3}\underset{\sim}{j}\right)+\left(-\frac{31}{3}\underset{\sim}{i}-\frac{22}{3}\underset{\sim}{j}+\frac{14}{3}\underset{\sim}{k}\right)=\frac{20}{3}\left(2\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}\right)-\frac{1}{3}\left(31\underset{\sim}{i}+22\underset{\sim}{j}-14\underset{\sim}{k}\right)$

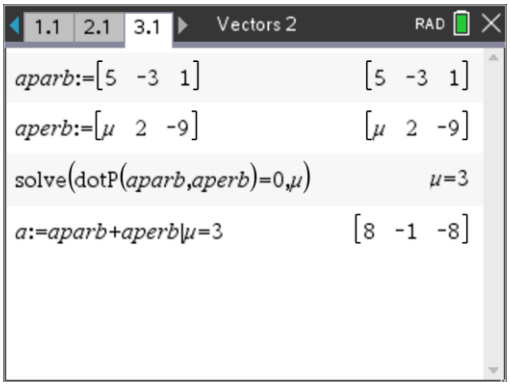


Note that the question specifies that $\underset{\sim}{v}$ must be stated as a sum, so find the resolutes individually and the answer is the sum of these resolutes

The vector resolute of $\underset{\sim}{v}$ parallel to $\underset{\sim}{w}$ is $\frac{\underset{\sim}{v}\cdot\underset{\sim}{w}}{|\underset{\sim}{w}|^2}\underset{\sim}{w}=\left(\underset{\sim}{v}\cdot\hat{\underset{\sim}{w}}\right)\hat{\underset{\sim}{w}}$ and the vector resolute of $\underset{\sim}{v}$ perpendicular to $\underset{\sim}{w}$ is $\underset{\sim}{v}-\left(\underset{\sim}{v}\cdot\hat{\underset{\sim}{w}}\right)\hat{\underset{\sim}{w}}$.

The answer may be verified by adding the final two results.

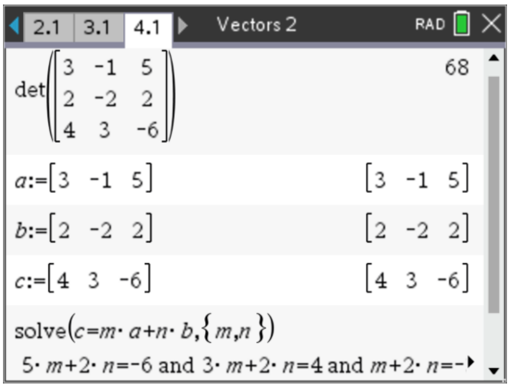
Question 3 $\mu=3;\ \underset{\sim}{a}=8\underset{\sim}{i}-\underset{\sim}{j}-8\underset{\sim}{k}$



The vector resolute of $\underset{\sim}{a}$ parallel to $\underset{\sim}{b}$ and the vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ are perpendicular, by definition. Therefore their dot product is 0. Solving this equation results in $\mu=3$

The vector $\underset{\sim}{a}$ is the sum of the two resolutes. Ensure that the condition $\mu=3$ is included (alternatively, μ can be defined to be 3).

Question 4 Yes, the vectors are linearly independent.



One way to test for linear dependence is to find the determinant of the 3×3 matrix that is constructed from the components of the vectors (each row represents a vector). If the determinant is 0, the vectors are linearly dependent. If the determinant is not 0, the vectors are linearly dependent.

Determinants can be found using the det command ([menu] [7] [3]).

Alternatively, the equation $\underset{\sim}{c}=m\underset{\sim}{a}+n\underset{\sim}{b}$ can be solved for m and n. Note that there is no solution to the simultaneous equations (∴ independence).

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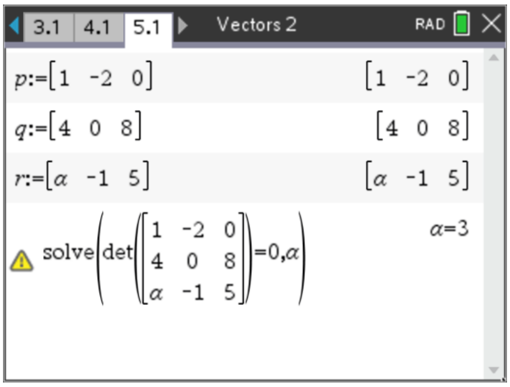
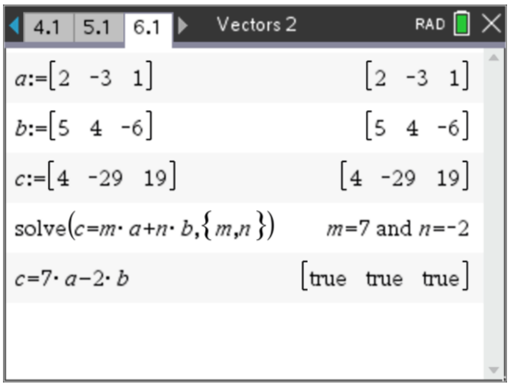
Question 5 $\alpha = 3$



Solving for the zeros of the determinant of the matrix comprised of the vector coefficients gives the value of α .

Question 6 $\underset{\sim}{c} = 7\underset{\sim}{a} - 2\underset{\sim}{b}$



Set $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$ (that is, $\underset{\sim}{c}$ is a linear combination of $\underset{\sim}{a}$ and $\underset{\sim}{b}$). Then solve this equation for m and n using the solve command.

The answer can be verified as shown.

Question 7

Consider a circle of radius r centred at O .

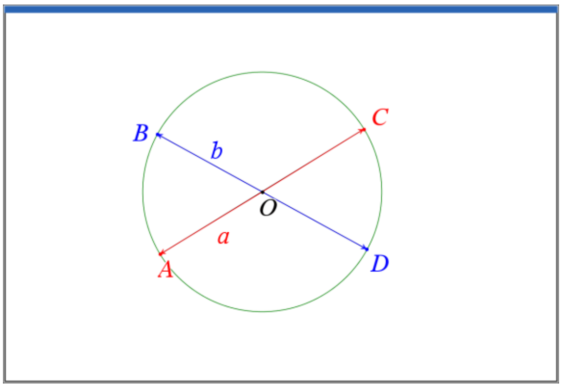
Let A be a point on the circumference such that $\overrightarrow{OA} = \underset{\sim}{a}$ is a radius.

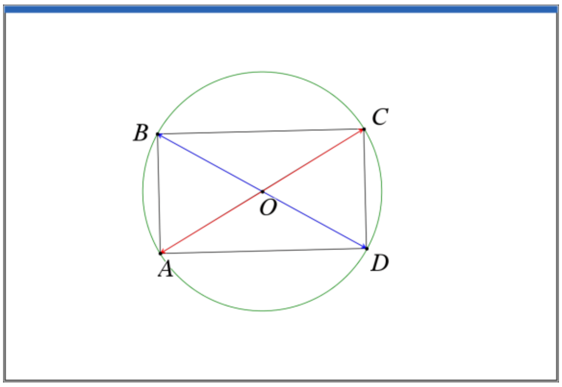
Let C be a point such that $\overrightarrow{AC}$ is a diameter. Therefore, $\overrightarrow{OC} = -\underset{\sim}{a}$.

Similarly, let B be a point on the circumference with $\overrightarrow{OB} = \underset{\sim}{b}$.

Let D be a point such that $\overrightarrow{BD}$ is a diameter. Then $\overrightarrow{OD} = -\underset{\sim}{b}$.

See the diagram to the left.





Finding $\overrightarrow{AB}$: $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\underset{\sim}{a} + \underset{\sim}{b}$

Finding $\overrightarrow{DC}$: $\overrightarrow{DC} = \overrightarrow{DO} + \overrightarrow{OC} = \underset{\sim}{b} - \underset{\sim}{a}$

☆ Therefore $\overrightarrow{AB} = \overrightarrow{DC}$.

Finding $\overrightarrow{AD}$: $\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD} = -\underset{\sim}{a} - \underset{\sim}{b}$

Finding $\overrightarrow{BC}$: $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\underset{\sim}{b} - \underset{\sim}{a}$

☆ Therefore $\overrightarrow{AD} = \overrightarrow{BC}$.

☆ The opposite sides of the quadrilateral $ABCD$ are parallel and each pair has the same length.

Finding the angle at vertex A:

$$\overrightarrow{AB} \cdot \overrightarrow{AD} = (-\underset{\sim}{a} + \underset{\sim}{b}) \cdot (-\underset{\sim}{a} - \underset{\sim}{b})$$

$$= \underset{\sim}{a} \cdot \underset{\sim}{a} + \cancel{\underset{\sim}{a} \cdot \underset{\sim}{b}} - \cancel{\underset{\sim}{a} \cdot \underset{\sim}{b}} - \underset{\sim}{b} \cdot \underset{\sim}{b}$$

$$= |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2 = 0, \text{ since } |\underset{\sim}{a}| = |\underset{\sim}{b}| = r$$

☆ Therefore $\overrightarrow{AB} \perp \overrightarrow{AD}$, so the quadrilateral is a rectangle.

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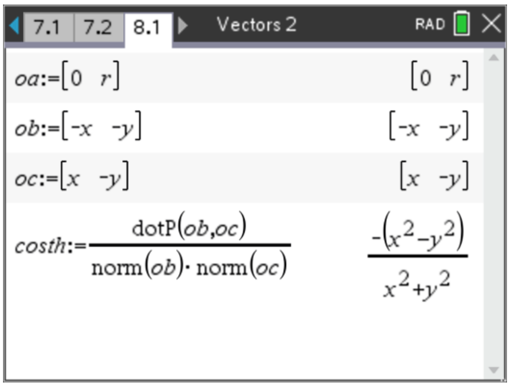
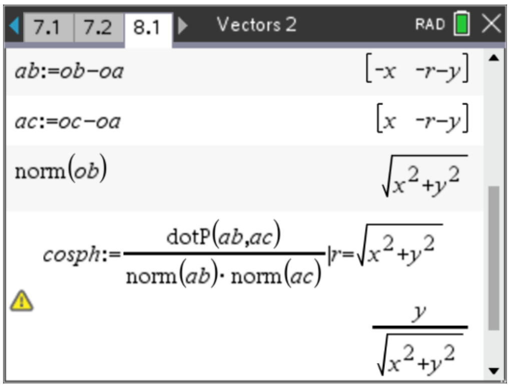
Question 8

$\overrightarrow{OA} = r\underset{\sim}{j}$, $\overrightarrow{OB} = -x\underset{\sim}{i} - y\underset{\sim}{j}$, $\overrightarrow{OC} = x\underset{\sim}{i} - y\underset{\sim}{j}$

$$\cos(\theta) = \frac{\overrightarrow{OB} \cdot \overrightarrow{OC}}{\left|\overrightarrow{OB}\right|\left|\overrightarrow{OC}\right|}$$

$$= \frac{y^2 - x^2}{x^2 + y^2}, \text{ as required}$$





$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $\quad\quad$ $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= -x\underset{\sim}{i} - (r+y)\underset{\sim}{j}$, $\quad\quad$ $= x\underset{\sim}{i} - (r+y)\underset{\sim}{j}$

Since $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are radii of the circle, $\left|\overrightarrow{OB}\right| = \left|\overrightarrow{OC}\right| = r$, so

$r = \sqrt{x^2 + y^2}$, using the norm command.

$$\cos(\varphi) = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$$

$$= \frac{y}{\sqrt{x^2 + y^2}}, \text{ as required}$$

To prove that $\theta = 2\varphi$, consider a cosine double angle formula for φ:

$$\cos(2\varphi) = 2\cos^2(\varphi) - 1$$

$$= 2\left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2 - 1$$

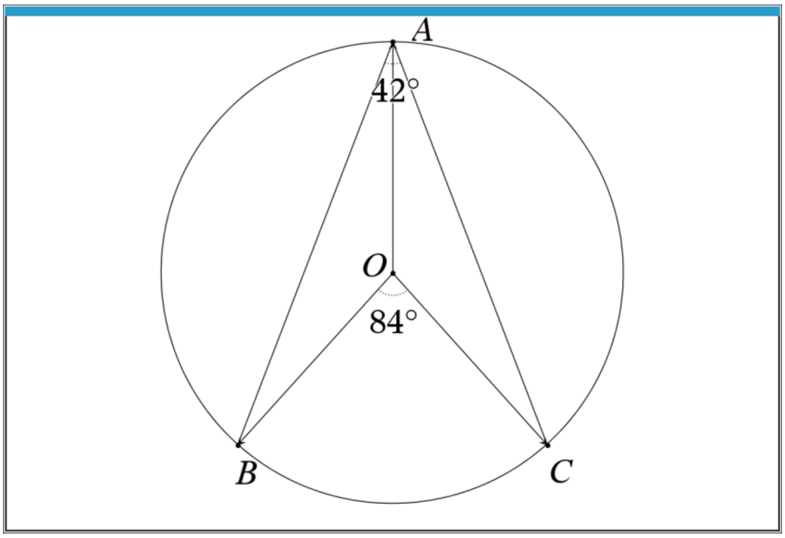
$$= \frac{2y^2}{x^2 + y^2} - 1$$

$$= \frac{y^2 - x^2}{x^2 + y^2}$$

$$= \cos(\theta)$$

$\cos(2\varphi) = \cos(\theta)$, so $\theta = 2\varphi$, as required.

The CAS can be used to demonstrate this property, and dynamic geometry can be used to verify this is true for all angles.



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