

Complex Numbers Part 2

Question: 1.

If $z = a \operatorname{cis}\left(\frac{\pi}{b}\right)$, $a, b \neq 0$, then $(\bar{z})^{-1}$ is equal to:

- A. $\frac{1}{a} \operatorname{cis}\left(\frac{\pi}{b}\right)$
- B. $\frac{1}{a} \operatorname{cis}\left(-\frac{\pi}{b}\right)$
- C. $\frac{1}{a} \operatorname{cis}\left(\frac{b}{\pi}\right)$
- D. $\frac{1}{a} \operatorname{cis}\left(-\frac{b}{\pi}\right)$
- E. $\bar{a} \operatorname{cis}\left(\frac{b}{\pi}\right)$

Question: 2.

If $z = -a + ai$ where $a > 0$ then $\operatorname{Arg}(z^3)$ is equal to:

- A. $\frac{27\pi^3}{64}$
- B. $\frac{\pi}{4}$
- C. $-\frac{\pi}{4}$
- D. $\frac{9\pi}{4}$
- E. $\frac{3\pi}{4}$

Question: 3.

If $w^2 = 16cis\left(\frac{\pi}{3}\right)$ then a possible value of w is:

- A. $4cis\left(\frac{\pi}{6}\right)$ B. $4cis\left(\frac{2\pi}{3}\right)$ C. $8cis\left(\frac{\pi}{6}\right)$ D. $16cis\left(\frac{\pi}{6}\right)$ E. $32cis\left(\frac{2\pi}{3}\right)$

Question: 4.

In the complex plane, the point $2 - i$ lies on the graph of the relation

- A. $\text{Arg}(z) = \frac{\pi}{6}$
B. $|z| = |z + 1|$
C. $2\text{Re}(z) = \text{Im}(z)$
D. $|z - 2| = 1$
E. $(\bar{z})^2 = 2z$

Question: 5.

Sets of points in the complex plane are defined by

$$S = \{z : |z + 1 - 2i| = 5\} \text{ and } T = \{z : \text{Re}(z) + 2\text{Im}(z) = 8\}.$$

Find the coordinates of the points of intersection between S and T .

Question: 6.

Find cube roots of $-27i$. Give your answer in the form $a + bi$, $a, b \in R$.

Question: 7.

Find the values of n for which: $(\sqrt{3} + i)^n - (\sqrt{3} - i)^n = 0$.

Question: 8.

Given that $z = (b + i)^2$, $b \in R^+$, find the value of b when $\text{Arg}(z) = \frac{\pi}{6}$.

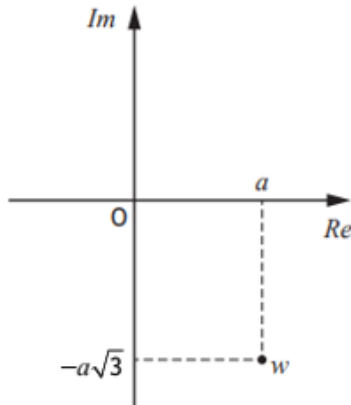
Question: 9.

Given that $|z| = 2\sqrt{5}$, find the complex number z that satisfies the equation

$$\frac{25}{z} - \frac{15}{\bar{z}} = 1 - 8i$$

Question: 10.

The complex number w has been plotted on an Argand diagram, as shown below.



where $a > 0$.

- a) Express w in Cartesian form and in polar form.

The complex number z_1 is a root of $z^3 = w$, where $z_1 = k \operatorname{cis}\left(\frac{\pi}{m}\right)$ for $k, m \in \mathbb{Z}$.

Given that $a = 4$,

- b) determine the values of k and m ,
c) find the remaining roots.

Answers

Question 1 Answer: A

Set your document in polar

The image shows two windows from a TI-84 Plus CE II calculator. The left window is the 'Document Settings' dialog box. The 'Real or Complex' dropdown menu is open, and 'Polar' is selected. Other settings include 'Display Digits: Float 6', 'Angle: Radian', 'Exponential Format: Normal', 'Calculation Mode: Real', and 'CAS Mode: Polar'. The right window is a document titled '*Doc' in 'RAD' mode. It shows the expression $z = e^{\frac{i \cdot \pi}{b} \cdot a}$ and its conjugate inverse $(\text{conj}(z))^{-1}$. The conjugate inverse is shown as $e^{-\frac{i \cdot \pi}{b} \cdot a}$, which is highlighted in yellow.

Question 2 Answer: B

The image shows a document window titled '*Doc' in 'RAD' mode. It displays the expression $z = -a + a \cdot i$ and its conjugate $-a + a \cdot i$. Below this, the angle of z^3 is calculated as $\text{angle}(z^3) | a > 0$, which is highlighted in yellow and shows the result $\frac{\pi}{4}$.

Question 3 Answer: A

The image shows a document window titled '*Doc' in 'RAD' mode. It displays the expression $z = 16 \cdot e^{\frac{\pi}{3} \cdot i}$ and its rectangular form $8 + 8 \cdot \sqrt{3} \cdot i$. Below this, the square root of z is calculated as $\sqrt{z} = 2 \cdot \sqrt{3} + 2 \cdot i$. The expression $(2 \cdot \sqrt{3} + 2 \cdot i) \blacktriangleright \text{Polar}$ is shown, which is highlighted in yellow and shows the result $e^{\frac{i \cdot \pi}{6}} \cdot 4$.

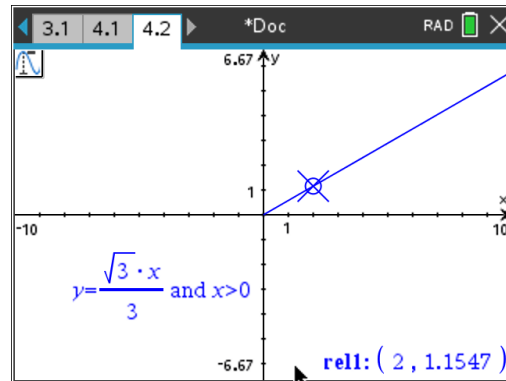
Question 4 **Answer: D**

Method 1

Draw a ray in answer A

$z = x + y \cdot i$ $x + y \cdot i$
 $\text{angle}(z)$ $\frac{\pi \cdot \text{sign}(y)}{2} - \tan^{-1}\left(\frac{x}{y}\right)$
 $\text{solve}\left(\frac{\pi \cdot \text{sign}(y)}{2} - \tan^{-1}\left(\frac{x}{y}\right) = \frac{\pi}{6}, y\right)$
 $y = \frac{\sqrt{3} \cdot x}{3}$ and $x > 0$

Draw in Relations and Trace, enter $x=2$



A is incorrect

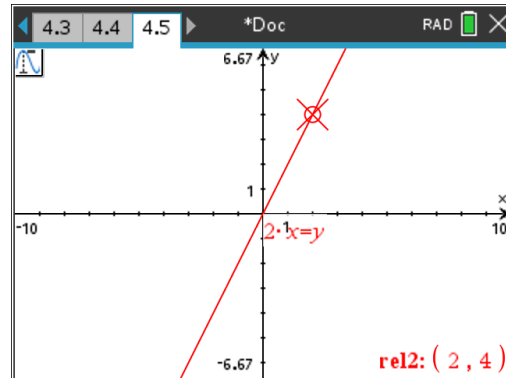
Check B (perpendicular bisector)

$(\sqrt{x^2 + y^2} = \sqrt{x^2 + 2 \cdot x + y^2 + 1})^2$
 $x^2 + y^2 = x^2 + 2 \cdot x + y^2 + 1$
 $(x^2 + y^2 = x^2 + 2 \cdot x + y^2 + 1) - x^2$ $y^2 = 2 \cdot x + y^2 + 1$
 $(y^2 = 2 \cdot x + y^2 + 1) - y^2$ $0 = 2 \cdot x + 1$
 $\text{solve}(0 = 2 \cdot x + 1, x)$ $x = \frac{-1}{2}$

Check C

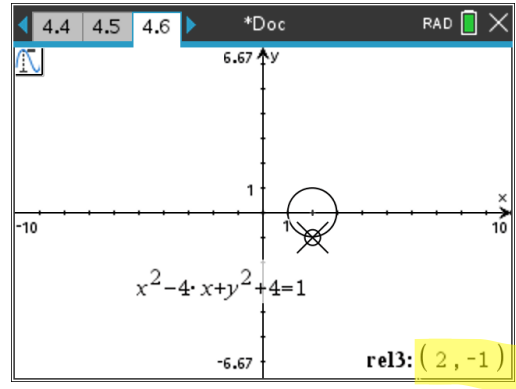
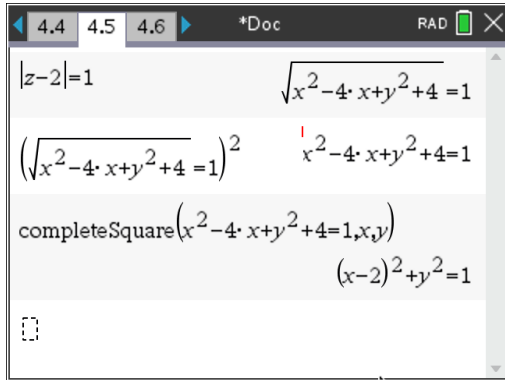
$z = x + y \cdot i$ $x + y \cdot i$
 $2 \cdot \text{real}(z) = \text{imag}(z)$ $2 \cdot x = y$

B is incorrect



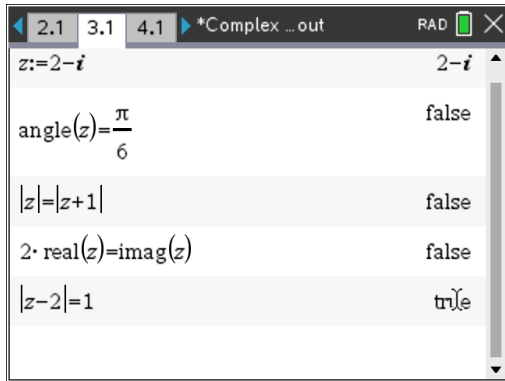
C is incorrect

Check D



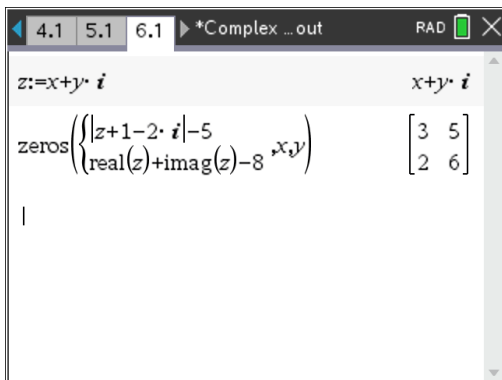
D is correct

Method 2



Question 5

(2, 6) and (3, 5)



Question 6

$$\frac{3\sqrt{3}}{2} - \frac{3}{2}i, 3i, -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

Find the first cube root. Start in polar form and convert to Rectangular

The calculator screen shows the following steps:

- Input: $-27 \cdot i$
- Conversion to polar form: $e^{i \cdot \frac{-\pi}{2}} \cdot 27$
- Exponentiation by $\frac{1}{3}$: $\left(e^{i \cdot \frac{-\pi}{2}} \cdot 27 \right)^{\frac{1}{3}}$
- Conversion to rectangular form: $\left(e^{i \cdot \frac{-\pi}{6}} \cdot 3 \right) \rightarrow \text{Rect}$
- Result: $\frac{3 \cdot \sqrt{3}}{2} - \frac{3}{2}i$ (highlighted in yellow)

Find the other two roots in polar form and convert to cartesian.

Second root:

The calculator screen shows the following steps:

- Angle calculation: $\frac{-\pi}{6} + \frac{2 \cdot \pi}{3}$
- Resulting angle: $\frac{\pi}{2}$
- Magnitude: $3 \cdot e^2$
- Result: $3 \cdot i$ (highlighted in yellow)

Third root:

The calculator screen shows the following steps:

- Angle calculation: $\frac{\pi}{2} + \frac{2 \cdot \pi}{3}$
- Resulting angle: $\frac{7 \cdot \pi}{6}$
- Magnitude: $3 \cdot e^2$
- Result: $3 \cdot e^{i \cdot \frac{7 \cdot \pi}{6}}$
- Conversion to rectangular form: $\left(e^{i \cdot \frac{7 \cdot \pi}{6}} \cdot 3 \right) \rightarrow \text{Rect}$
- Result: $-\frac{3 \cdot \sqrt{3}}{2} - \frac{3}{2}i$ (highlighted in yellow)

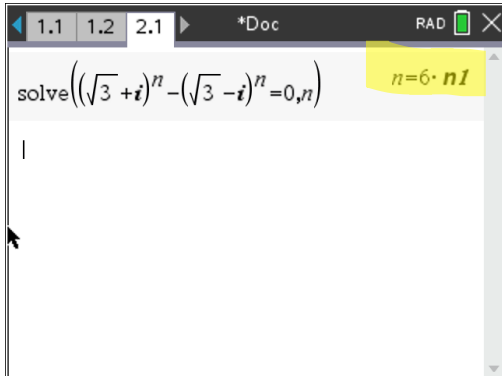
Check:

The calculator screen shows the following steps:

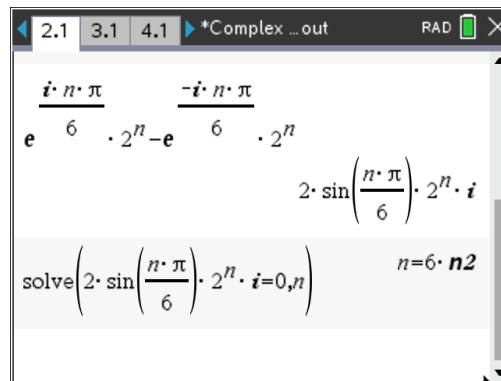
- Verification of the first root: $\left(\frac{-3 \cdot \sqrt{3}}{2} - \frac{3}{2}i \right)^3 = -27 \cdot i$
- Verification of the second root: $(3 \cdot i)^3 = -27 \cdot i$
- Verification of the third root: $\left(\frac{3 \cdot \sqrt{3}}{2} - \frac{3}{2}i \right)^3 = -27 \cdot i$

Question 7

$$n = 6k, k \in \mathbb{Z}.$$



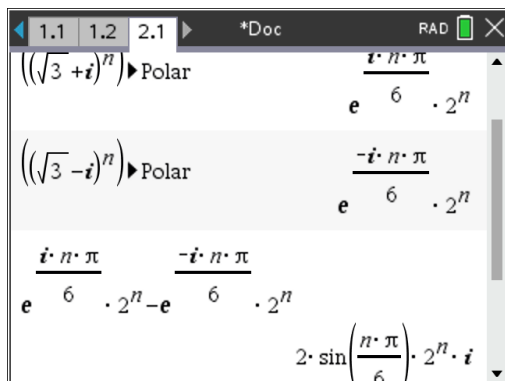
Deduce that it only has an imaginary part, real parts cancel out. Therefore, the equation is zero when



which gives us the same answer as a general solution to a trig equation.

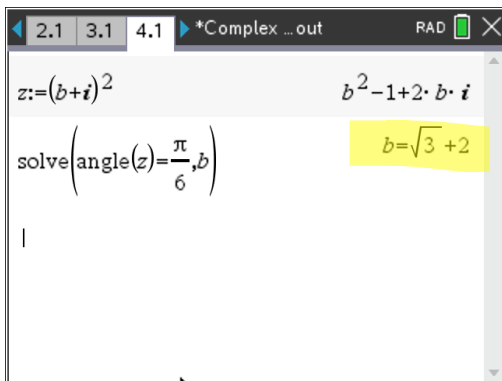
Alternatively:

Convert to polar first, subtract to see the result.



Question 8

$$b = 2 + \sqrt{3}$$

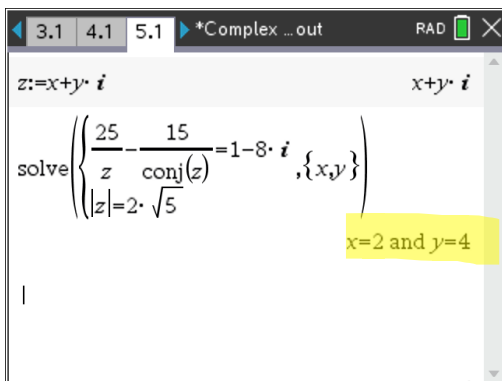


A screenshot of a TI-84 Plus calculator window titled '*Complex ...out' in RAD mode. The window shows the following content:

- Input: $z := (b+i)^2$
- Output: $b^2 - 1 + 2 \cdot b \cdot i$
- Input: $\text{solve}\left(\text{angle}(z) = \frac{\pi}{6}, b\right)$
- Output: $b = \sqrt{3} + 2$ (highlighted in yellow)

Question 9

$$z = 2 + 4i$$



A screenshot of a TI-84 Plus calculator window titled '*Complex ...out' in RAD mode. The window shows the following content:

- Input: $z := x + y \cdot i$
- Output: $x + y \cdot i$
- Input: $\text{solve}\left(\left\{\begin{array}{l} \frac{25}{z} - \frac{15}{\text{conj}(z)} = 1 - 8 \cdot i \\ |z| = 2 \cdot \sqrt{5} \end{array}\right\}, \{x, y\}\right)$
- Output: $x = 2 \text{ and } y = 4$ (highlighted in yellow)

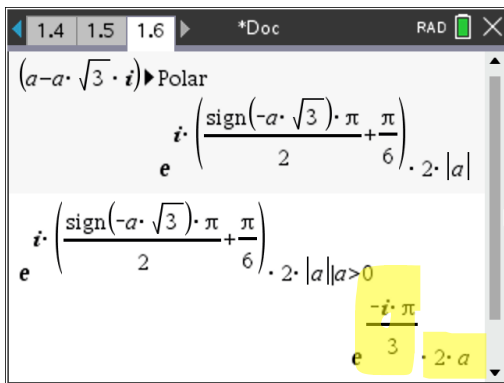
Question 10

a) $w = a - a\sqrt{3}i; w = 2a \operatorname{cis}\left(-\frac{\pi}{3}\right).$

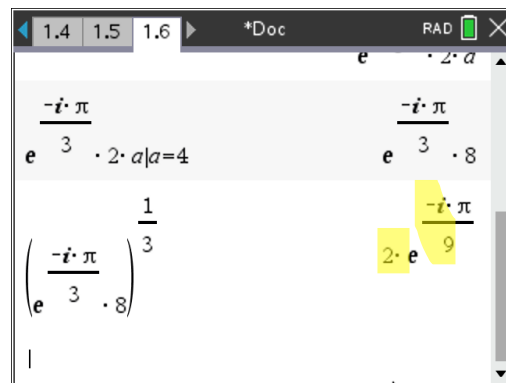
b) $k = 2, m = -9$

c) $z_2 = 2 \operatorname{cis}\left(\frac{5\pi}{9}\right), z_3 = 2 \operatorname{cis}\left(-\frac{7\pi}{9}\right)$

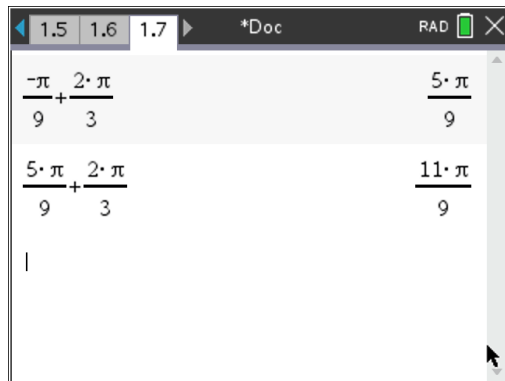
a) Convert to polar with condition $a > 0$.



b) With CAS in polar setting, set $a = 4$ and then raise the obtained answer to the power of $1/3$.



c) The remaining roots are evenly spread over the circle with radius 2 every $\frac{2\pi}{3}$:



and express with a principal argument.