Compound Interest

Math Objectives

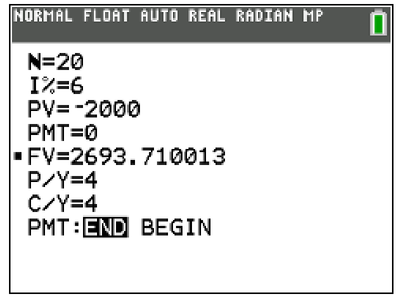
- Students will interpret the variables in the formula for compound interest.
- Students will use the formula for compound interest and understand the effects of changes in the interest rate and the number of compounding periods.
- Students will understand the relationship between compound interest and continuous compounding.
- Model with mathematics (CCSS Mathematical Practice).

Vocabulary

- compound interest
- interest rate
- pay periods
- initial deposit
- continuous compounding

About the Lesson

- This lesson is aligning with the curriculum of IB Mathematics Applications and Interpretations SL/HL and IB Mathematics Approaches and Analysis SL/HL
- This falls under the IB Mathematics Core Content Topic 1 Number and Algebra:
  1.4 Financial applications of geometric sequences and series involving compound interest and annual depreciation.
- This lesson involves exploring the formula for compound interest as a function of the initial deposit, interest rate, and the number of pay periods per year.
- As a result, students will:
  - Learn the relationship between the interest rate and the total amount in the account.
  - Learn the relationship between the number of pay periods and the total amount in the account.
  - Discover the limiting condition as the number of pay periods increases without bound.



Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions given within may be required if using other calculator models.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Materials:
Student Activity
Compound_Interest_Student-84.pdf
Compound_Interest_Student-84.doc

Teacher Preparation and Notes

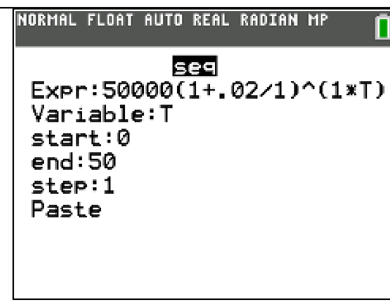
- Students should be familiar with creating lists on the TI-84 Plus Family of devices by inputting formulas, such as **seq(**.

Activity Materials

- Compatible TI Technologies:
  TI-84 Plus*, TI-84 Plus Silver Edition*, TI-84 Plus C Silver Edition, TI-84 Plus CE

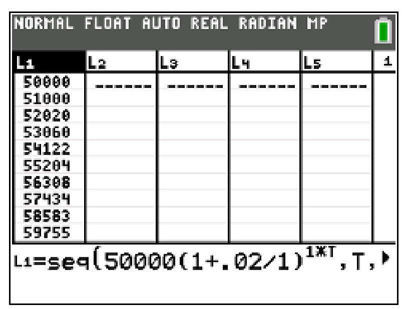
Discussion Points and Possible Answers

> **Teacher Tip:** When using the compound interest formula, some international students may recognize it in an alternate form written as
>
> $$FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$$
>
> where *FV* is the future value, *PV* is the present value, *k* is the number of compounding periods per year, and *r%* is the annual rate of interest.



Let P be the initial amount (**Principal**) deposited, r the annual interest rate expressed as a decimal, n the number of times interest is paid each year, and A the total amount in the account at time t (in years). The formula for compound interest is

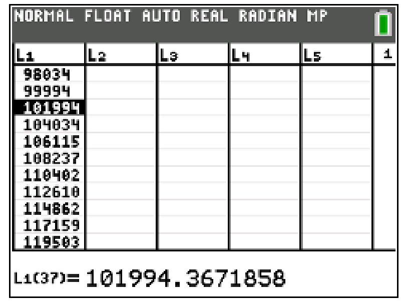
$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$



1. Suppose $50,000 is deposited in an account paying 2% ($r = 0.02$) per year ($n = 1$). On your handheld, press **Stat > Edit**, place your cursor at the top of **L$_1$** and press Enter. Now press **2nd > Stat > Ops > 5: seq(**. You will have to enter the following: expression (formula), variable (T), start (0), end (50), and step (1). This will give you the total amount for each of the first 50 years of the investment.

   a. If you subtract each total and its previous total (such as year 2 minus year 1), you will find the interest earned each year. Explain why the interest earned after each pay period increases.

Answer: After each pay period, the account balance is the original deposit, or principal, plus interest. Therefore, interest is paid based on a larger account balance each pay period.



b. Using your table, estimate the number of years until the initial deposit doubles.

Answer: The initial deposit doubles after 36 years. Row 37 of the spreadsheet indicates the total amount in the account is $101,994.37.

> **Teacher Tip:** Students might suggest the initial deposit doubles between year 35 and year 36. However, remember that interest is only paid once per year ($n = 1$). We assume no additional interest is earned until the end of the pay period.

c. Find the interest rate so that the initial deposit doubles after 15 years.

Answer: For $r = 0.0473$ (interest rate of 4.73%, approximately 5%), the initial deposit will double after 15 years. Note: Student answers will vary. Consider asking for the smallest interest rate such that the initial deposit doubles after 15 years. Consider asking for an interest rate so that the initial deposit doubles after 15 years, but no earlier.

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2. Suppose $10,000 is deposited in an account paying 5% ($r = 0.05$) semi-annually ($n = 2$).

a. Complete the following table to find the amount in the account after two years.

Answer:

n	2	4	6	12	52
$A(2)$	11,038.13	11,044.86	11,047.13	11,049.41	11,051.18

As n increases, explain how you would expect the value of $A(t)$ to change for a fixed value of t.

Answer: For a fixed value of t the table suggests that as n increases, the amount in the account at time t, $A(t)$, also increases.

b. Explain the meaning of each of the following:

$n = 365$;
$n = (365)(24) = 8760$;
$n = (365)(24)(60) = 525{,}600$; and
$n = (365)(24)(60)(60) = 31{,}536{,}000$.

Answer:

$n = 365$: Interest is paid daily.
$n = 8760$: Interest is paid hourly.
$n = 525{,}600$: Interest is paid every minute.
$n = 31{,}536{,}000$: Interest is paid every second.

c. Complete the following table.

n	365	8760	525,600	31,536,000
$A(2)$	11,051.63	11,051.71	11,051.71	11,051.72

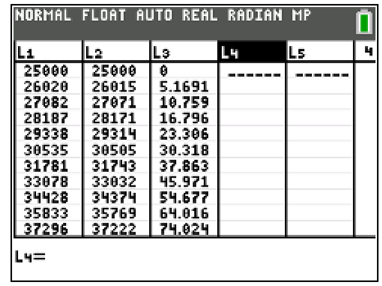
d. As n increases, describe the compounding period. Explain how the amount in the account changes for a fixed value of t as n increases.

Answer: As n increases, the number of compounding periods increases, towards interest being paid continuously, or continuous compounding. This question suggests that as n increases, the amount in the account at time t, $A(t)$, also increases.

e. Using your results from Questions 1 and 2, describe the characteristics you would like in an account in order to earn the most interest after every pay period.

Answer: In order to earn the most in an account after every pay period, we should search for the greatest interest rate and an account with the greatest number of pay periods.

3. Suppose \$25,000 is deposited in an account paying 4% ($r = 0.04$) quarterly ($n = 4$). In L_2, enter this information as you did in Problem 1, this will display the amount in the account, A, after each pay period. L_1 contains values of the function $c(t) = Pe^{rt}$ for each corresponding pay period, where $e \approx 2.71828\ldots$, the base of the natural logarithm. This function does not depend upon *n* (number of compounding periods per year) as it is the compounded continuously formula. In L_3, find the difference between the two values for corresponding pay periods by subtracting $L_1 - L_2$.



L1	L2	L3	L4	L5
25000	25000	0	------	------
26020	26015	5.1691		
27082	27071	10.759		
28187	28171	16.796		
29338	29314	23.306		
30535	30505	30.318		
31781	31743	37.863		
33078	33032	45.971		
34428	34374	54.677		
35833	35769	64.016		
37296	37222	74.024		

As n increases, explain the relationship between $c(t)$ (L_1) and $A(t)$ (L_2).

<u>Answer:</u> As n increases, the values of $A(t)$ tend to get closer and closer to $c(t)$, but $A(t) \le c(t)$ for all values of t.

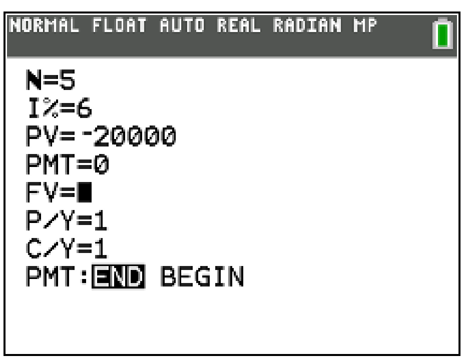
Using the Finance Solver on the handheld:

Insert a calculator page. Press **Apps < 1: Finance, < 1: TVM Solver**. The TVM Solver page will open for you to use in place of the compound interest formula used earlier in this activity.

Sample:

Find the future value of a \$20,000 invested for 5 years at 6% compounded annually.

This is what it should look like on the handheld:



Please notice that the **PV** (Principal/Present Value) is entered as -20000 because cash outflows are considered negative. Place your cursor over **FV** and press enter to find the Future Value.

FV = \$26,764.51

4. Find the future value of $2000 invested for 5 years at 6% compounded quarterly.

Answer: $2,693.71

Note: There are two ways to input values in the Solver. You can input **N = 5**, $\boldsymbol{P/Y=1}$, and $\boldsymbol{C/Y=4}$, or input **N = 4 · 5**, $\boldsymbol{P/Y=4}$, and $\boldsymbol{C/Y=4}$.

5. Find the value of $8000 invested for 6 years at 8% compounded monthly.

Answer: $12,908.02

6. Find how much you would have to invest in a savings account paying 6% compounded quarterly in order to have $3000 in 5 years.

Answer: $2,227.41 (this will be negative on the handheld because it is paid out by the investor)

Wrap Up

Upon completion of this activity, students should be able to understand:

- The relationship between the interest rate and the total amount in the account.
- The relationship between the number of pay periods and the total amount in the account.
- How to find the amount of an investment by hand and by using the Finance Solver.
- The limiting condition of compound interest as the number of pay periods increases without bound.
- A very basic idea of a limit.
- A very basic idea of continuous compounding, or interest being paid at every instant.

Teacher Notes

1. The graph of $A(t)$ is presented as a smooth curve. In practice, $A(t)$ is a piecewise linear function since interest is paid a discrete periods. Consider a graph of the calculator function $ap(t) = P\left(1+\frac{r}{n}\right)^{\text{floor}(nt)}$
2. For any fixed value of t, for example t_0, the value $c(t_0) = Pe^{rt_0}$ is the limit of $A(t_0)$ as n increases. This presents a good opportunity for students to discover the idea of a limit.
3. Suppose the initial deposit is $1 and the interest rate is 100% ($r = 1$). At the end of 1 year, the amount in the account is $A(1) = \left(1+\frac{1}{n}\right)^n$. Ask students to construct a table of values for $A(1)$ for various values of n. For example:

n	$\left(1+\frac{1}{n}\right)^n$
1	2.000000
5	2.488320
10	2.593742
100	2.704814
1000	2.716924
10,000	2.718146
100,000	2.718268
1,000,000	2.718280

*This table might help suggest why the number e is associated with compound interest and appears in the formula for $c(t)$.