Astroid

Student Activity

7 8 9 10 11 **12**

TI-Nspire™ Investigation Student 30 min

Introduction

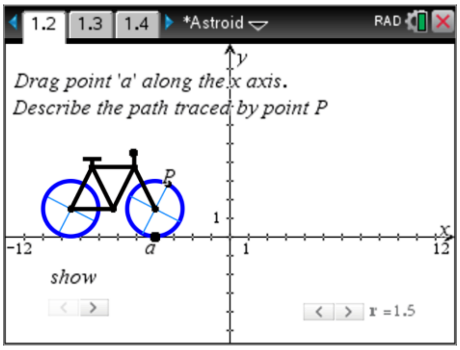
How is the motion of a ladder sliding down a wall related to the motion of the valve on a bicycle wheel or to a popular amusement park ride? A little mathematical history, some parametric equations and calculus hold the key. Galileo first studied the cycloid in 1599. A cycloid describes the motion of the valve on a bicycle wheel. (Astroid TNS File: Page 1.2) Roemer (1674) put the cycloid into a 'spin' and developed the Astroid, not to be not to be confused with the celestial asteroid or Orbiter amusement park ride. The Astroid is a hypocycloid[1].

Cycloid

Open the TI-Nspire file: "Astroid". Navigate to page 1.2 of the file. Drag point A on the rim of the bicycle wheel and observe point P on the rim.

To see the trace (locus) of this point click on the 'show' button.

The size of the bicycle (wheel radius) can also be changed.

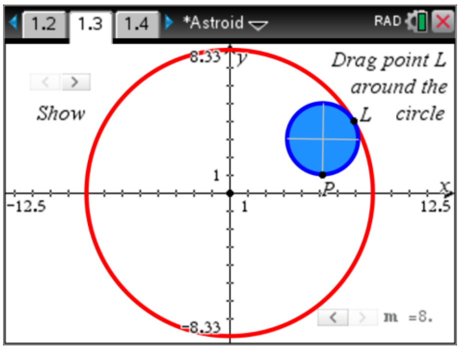


Astroid

Navigate to page 1.3.

Drag point L around the circle and watch the motion of point P.

To reveal the locus (path) of point P use the hide/show toggle.



Question: 1.

For each of the following questions: point L is moving counter-clockwise around the larger circle, the larger circle has a radius m measuring 8 units, the smaller circle has a radius of 2 units.

a. Write down the parametric equations for the motion of the **centre** of the smaller circle.

$x(t) = 6\cos(t)$

$y(t) = 6\sin(t)$

b. How many times does the smaller circle rotate as it travels around the inside of the larger circle?

The circle rotates just 3 times even though the circumference is ¼ of the larger circle.

---

[1] Hypo – means beneath, in this mathematical context it refers to a subset. So hypocycloid is a smaller set of the family of cycloids.

Author: P. Fox

Teacher Notes:

This is because one rotation of the smaller circle is 'undone' as it moves counter-clockwise around the larger circle. Students that attempt this question without actually practicing it on the diagram are likely to respond with 4 rotations.

c. Consider the small circle as simply rotating on its own axis, located at the centre of the diagram; determine the parametric equations for the motion of point P.
Make sure the equations also account for the direction and number of rotations.

$x(t) = 2\cos(3t)$
$y(t) = -2\sin(3t)$

d. Combine your answers to part A and C and see if they follow the movement of Point P as it travels to form the Astroid.

$x(t) = 6\cos(t) + 2\cos(3t)$
$y(t) = 6\sin(t) - 2\sin(3t)$

Question: 2.

The parametric equations established in the previous question can be simplified.

a. Show that $\cos(3t) = 4\cos^3(t) - 3\cos(t)$ and hence show that the parametric equation $x(t)$ for the Astroid can be written as: $x(t) = 8\cos^3(t)$

Methods will vary:

$$\begin{aligned}\cos(3t) &= \cos(2t)\cos(t) - \sin(2t)\sin(t)\\ &= \left(2\cos^2(t) - 1\right)\cos(t) - 2\sin(t)\cos(t)\sin(t)\\ &= 2\cos^3(t) - \cos(t) - 2\sin^2(t)\cos(t)\\ &= 2\cos^3(t) - \cos(t) - 2\cos(t)\left(1 - \cos^2(t)\right)\\ &= 4\cos^3(t) - 3\cos(t)\end{aligned}$$

Hence:

$$\begin{aligned}6\cos(t) + 2\cos(3t) &= 6\cos(t) + 2\left(4\cos^3(t) - 3\cos(t)\right)\\ &= 6\cos(t) + 8\cos^3(t) - 6\cos(t)\\ &= 8\cos^3(t)\end{aligned}$$

b. Show that $\sin(3t) = 3\sin(t) - 4\sin^3(t)$ and hence show that the parametric equation for $y(t) = 8\sin^3(t)$

Methods will vary:

$$\begin{aligned}\sin(3t) &= \cos(2t)\sin(t) + \sin(2t)\cos(t)\\ &= \left(1 - 2\sin^3(t)\right)\sin(t) + 2\sin(t)\cos^2(t)\\ &= \sin(t) - 2\sin^3(t) + 2\sin(t)\left(1 - \sin^2(t)\right)\\ &= 3\sin(t) - 4\sin^3(t)\end{aligned}$$

Hence:

$$\begin{aligned}6\sin(t)-2\sin^3(t)&=6\sin(t)-2\left(3\sin(t)-4\sin^3(t)\right)\\&=6\sin(t)-6\sin(t)+8\sin^3(t)\\&=8\sin^3(t)\end{aligned}$$

Question: 3.

Use the parametric equations for the Astroid to show that an equivalent Cartesian equation can be expressed as: $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ and that for this specific case: $a=8$.

$$\begin{aligned}x^{\frac{2}{3}}+y^{\frac{2}{3}}&=\left(8\cos^3(t)\right)^{\frac{2}{3}}+\left(8\sin^3(t)\right)^{\frac{2}{3}}\\&=4\cos^2(t)+4\sin^2(t)\\&=4\\&=8^{\frac{2}{3}}\end{aligned}$$

Question: 4.

Use implicit differentiation techniques to determine the gradient of the Astroid at any point.

$$\frac{d\left(x^{2/3}\right)}{dx}+\frac{d\left(y^{2/3}\right)}{dx}=\frac{d(8)}{dx}$$

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{d\left(y^{2/3}\right)}{dy}\frac{dy}{dx}=0$$

$$\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=-\frac{2}{3}x^{-\frac{1}{3}}$$

$$\frac{dy}{dx}=-\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}$$

$$\frac{dy}{dx}=-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

Question: 5.

Use the parametric equations for the Astroid to determine the gradient in terms of *t*.

$$x=8\cos^3(t) \qquad \frac{dx}{dt}=-24\cos^2(t)\sin(t)$$

$$y=8\sin^3(t) \qquad \frac{dx}{dt}=24\sin^2(t)\cos(t)$$

$$\frac{dy}{dx}=\frac{dy}{dt}\,\frac{dt}{dx}$$

$$\frac{dy}{dx}=-\frac{24\sin^2(t)\cos(t)}{24\cos^2(t)\sin(t)}$$

$$\frac{dy}{dx}=-\frac{\sin(t)}{\cos(t)}\quad\text{or}\quad-\tan(t)$$

Author: P. Fox

Question: 6.

Show that the two expressions for the derivative (Cartesian and Parametric) are equivalent.

$$x = 8\cos^3(t) \quad \text{and} \quad y = 8\sin^3(t)$$

$$\frac{x^{1/3}}{4} = \cos(t) \quad \text{and} \quad \frac{y^{1/3}}{4} = \sin(t)$$

$$\frac{dy}{dx} = -\frac{\sin(t)}{\cos(t)} \quad \text{by substitution ...}$$

$$\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

Question: 7.

Determine the equation to the tangent for: $t = \frac{\pi}{6}$, the corresponding x and y intercepts, the length of the tangent joining the intercepts and the angle the tangent makes with the positive x axis.

When: $t = \frac{\pi}{6}$	Tangent	Intercepts:	Length:	Angle:
$\frac{dy}{dx} = -\tan\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$	$y = -\frac{\sqrt{3}}{3}x + 4$	$(4\sqrt{3}, 0)$ $(0, 4)$	8	$\frac{5\pi}{6}$ or $-\frac{\pi}{6}$

Question: 8.

Determine the equation to the tangent for: $t = \frac{\pi}{4}$, the corresponding x and y intercepts, the length of the tangent joining the intercepts and the angle the tangent makes with the positive x axis.

When: $t = \frac{\pi}{4}$	Tangent	Intercepts:	Length:	Angle:
$\frac{dy}{dx} = -\tan\left(\frac{\pi}{4}\right) = -1$	$y = -x + 4\sqrt{2}$	$(4\sqrt{2}, 0)$ $(0, 4\sqrt{2})$	8	$\frac{3\pi}{4}$ or $-\frac{\pi}{4}$

Question: 9.

Determine the equation to the tangent for: $t = \frac{\pi}{3}$, the corresponding x and y intercepts, the length of the tangent joining the intercepts and the angle the tangent makes with the positive x axis.

When: $t = \frac{\pi}{3}$	Tangent	Intercepts:	Length:	Angle:
$\frac{dy}{dx} = -\tan\left(\frac{\pi}{4}\right) = -1$	$y = -\sqrt{3}x + 4\sqrt{3}$	$(4, 0)$ $(0, 4\sqrt{3})$	8	$\frac{2\pi}{3}$ or $-\frac{\pi}{3}$

Question: 10.

Comment on the angle the tangent makes with the x axis and your answer to question 5.

The angle the tangent makes with the positive x axis is given by – *t*. (As shown by the derivative –tan(*t*))

Question: 11.

Comment on the distance between the axes intercepts for each tangent equation.

The distance between the two axes intercepts for each tangent remains constant.

Question: 12.

Use integral calculus to determine the distance that point P moves as it travels along the length of the Astroid.

$$\text{From Previous Question: } \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\text{Arc Length} = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^8 \sqrt{1+\left(-\frac{y^{1/3}}{x^{1/3}}\right)^2}\,dx$$

$$= \int_0^8 \sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\,dx \quad \text{but } y^{\frac{2}{3}} = 4 - x^{\frac{2}{3}}$$

$$= \int_0^8 \sqrt{1+\frac{4-x^{2/3}}{x^{2/3}}}\,dx$$

$$= 2\int_0^8 \sqrt{\frac{1}{x^{2/3}}}\,dx$$

$$= 2\int_0^8 x^{-\frac{1}{3}}\,dx$$

$$= 12$$

Note that this is only the arc length for quadrant 1, therefore total distance travelled by point P = 48 units.

Teacher Notes:

Students should be encouraged to consider ways to check their answer, particularly given how easy it is to establish an estimate. The total length of the Astroid should be close to the circumference of the circle: $2\pi r \approx 50.265$. The total distance travelled by point P is slightly less than the circumference of the circle, this is because the curvature of the Astroid is less than that of the circle. This can be shown by reflecting a point on the circle across a line segment joining (8, 0) and (0, 8).

Author: P. Fox

Falling Ladder

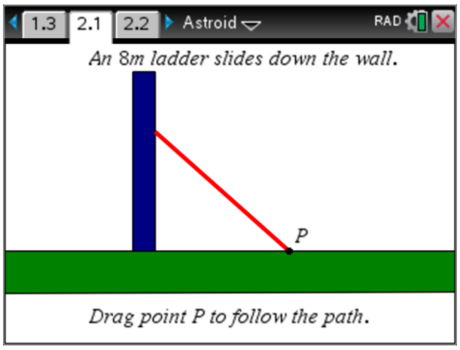
Navigate to page 2.1.

Drag point P around the ground and watch the motion of the ladder.

From the menu select:

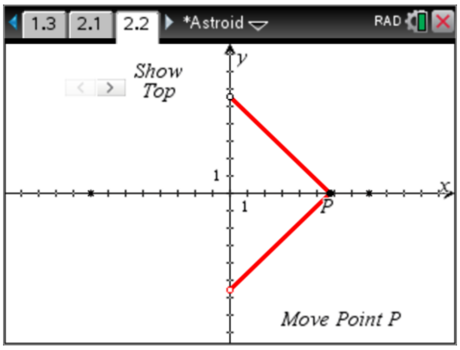
Geometry > **Construction** > **Locus**

Select the ladder followed by point P.



Navigate to page 2.2

Drag point P along the x axis. Use the toggle to show a trace of the ladder's position both above and below the x axis.



Exploration

Explore the motion of the ladder. Show that the *envelope* traced out by the movement of the ladder forms an Astroid?

There are numerous ways students may attempt this problem, using CAS will certainly help for many of them!

Example:

Consider the 'ladder' as forming a family of straight lines. The relationship between the x intercept and the y intercept is established by the constant length of the ladder, in this case 8 units. Let *t* be a parameter that represents the equation number in the family of straight lines. The objective is to determine the curve where this family of straight lines intersect (the envelope).

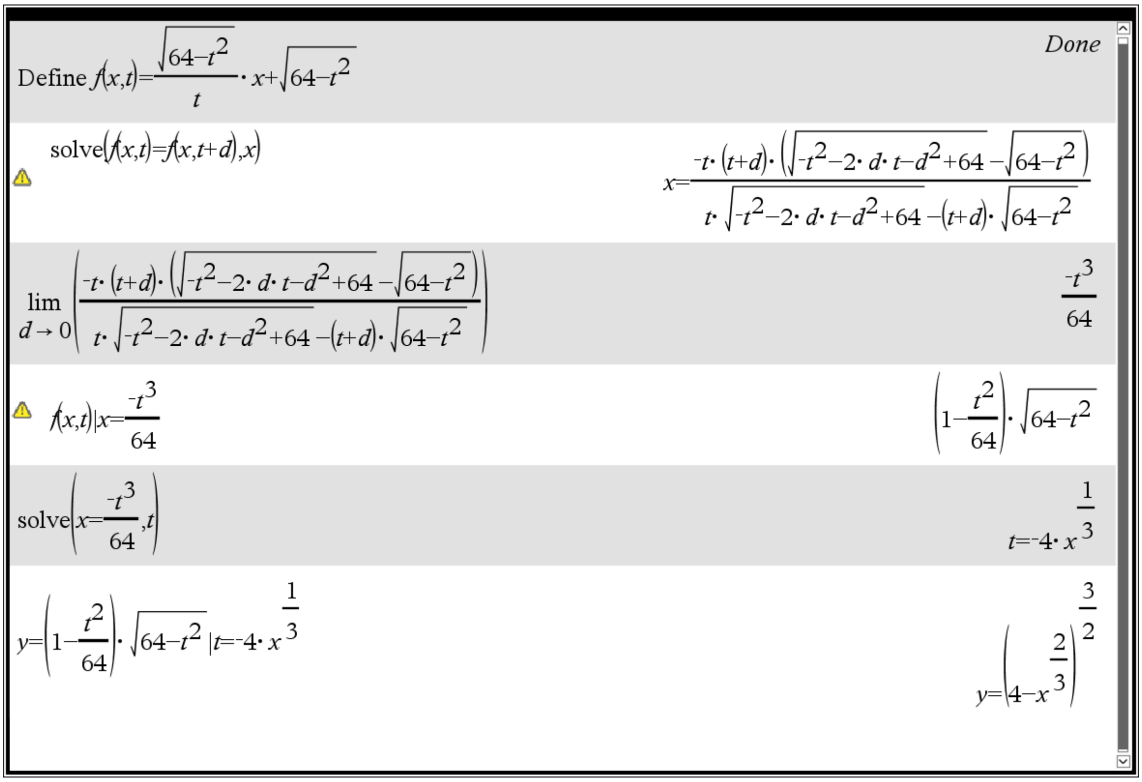
Gradient: $\frac{\sqrt{64-t^2}}{t}$

Equation: $f(x,t)=\frac{\sqrt{64-t^2}}{t}x+\sqrt{64-t^2}$

It is possible using the 'solve' command to determine the point of intersection between consecutive equations using: $solve(f(x,t)=f(x,t+1),x)$ however this assumes each line (ladder) is one unit apart.

An alternative is to consider: $solve(f(x,t)=f(x,t+d),x)$ and consider the limit as $d\to 0$.

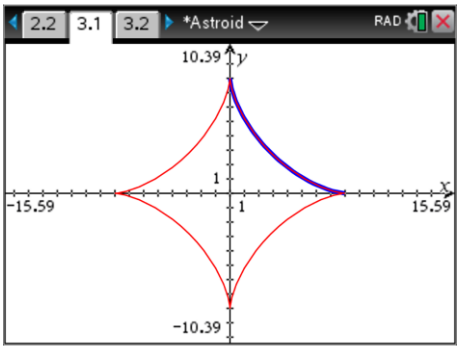
Author: P. Fox



The parametric equations can be graphed on the same graph as the Astroid: (See below)

$$x(t)=\frac{t^3}{64}$$

$$y(t)=\left(\frac{1-t^2}{64}\right)\sqrt{64-t^2}$$



Alternatively, by transposing the equation we see that $x^{\frac{2}{3}}+y^{\frac{2}{3}}=4$ or $x^{\frac{2}{3}}+y^{\frac{2}{3}}=8^{\frac{2}{3}}$

Author: P. Fox