

A Square Peg in a Round Hole



Student Activity

7 8 9 10 **11** 12



TI-Nspire



Investigation



Student



45 min

Introduction

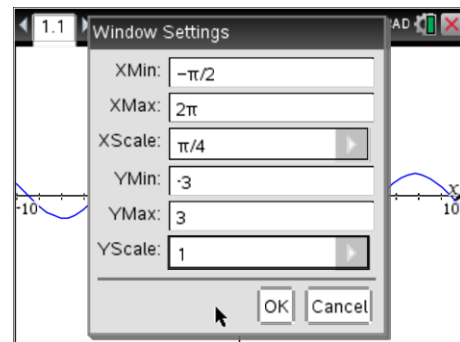
Using trigonometric functions such as $\sin(x)$ on a calculator is relatively easy, enter an angle and the ratio is produced immediately and generally accurate to at least ten decimal places. With the exception of a small group of ratios involving exact values, it is generally only the first four decimal places that are relevant to most calculations. In years gone by, prior to the wide spread use of calculators, trigonometric tables only provided such accuracy. This investigation looks at approximating trigonometric functions with a polynomial, this may seem like trying to fit a square peg in a round whole, but the results can be quite impressive!

A Quadratic Model

Start a new TI-nspire document, insert a Graphs application and graph the function:

$$f(x) = \sin(x)$$

Change the window settings to match those shown. Make sure document settings are set to radians.



Operating system 4.0 onwards provides for exact value specification in the Window Settings. For some earlier operating systems exact values can be achieved by entering them directly on the Graph application.

Question: 1.

Determine exact solutions to $\sin(x) = 0$ over the domain: $0 \leq x < 2\pi$.

Question: 2.

The quadratic function: $g(x) = (x-m)(x-n)$ passes through the same x intercepts as the function in question 1. Determine the values of m and n .

Question: 3.

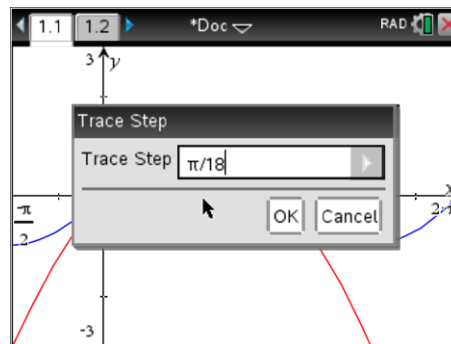
The quadratic function is further modified: $g(x) = a(x-m)(x-n)$. The function passes through the same x intercepts and also shares the same maximum value as: $\sin(x)$. Determine the exact value of a and graph the function: $g(x)$ and $\sin(x)$ on the same set of axis.

Press the [menu] key, select **Trace** followed by **Trace Step**.

Set the trace step to: $\frac{\pi}{18}$.

Return to the trace menu and select **Trace All**.

This will produce a vertical line passing through all the defined functions. Use the right or left arrow to systematically explore the values produced by $\sin(x)$ and $g(x)$.



Question: 4.

Complete the table of values below, correct to TWO decimal places. Comment on the accuracy of $g(x)$ to produce values for $\sin(x)$.

x	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{4\pi}{9}$	$\frac{\pi}{2}$
$\sin(x)$										
$g(x)$										

Question: 5.

Explain why it is only necessary (and appropriate) to explore the differences over the domain: $0 \leq x \leq \frac{\pi}{2}$.

Question: 6.

Explain why a cubic function may be less appropriate to model $\sin(x)$ over the domain: $0 \leq x \leq \pi$ but more appropriate over the domain: $0 \leq x \leq 2\pi$.

Considering $\sin(x)$ has more than two x intercepts, it follows that increasing the degree of the polynomial is required to provide an improved model.

Consider a quartic function of the form: $h(x) = a(x-m)(x-n)(x-p)(x-q)$ that is used to model $\sin(x)$ over the domain: $-\pi \leq x \leq 2\pi$.

Question: 7.

Given $h(x)$ and $\sin(x)$ share the same x intercepts over this domain and $h\left(\frac{\pi}{2}\right) = 1$, determine the rule for $h(x)$.

Question: 8.

Complete the table of values below, correct to THREE decimal places. Comment on the accuracy of $h(x)$ to produce values for $\sin(x)$ and compare this to predictions using $g(x)$.

x	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{4\pi}{9}$	$\frac{\pi}{2}$
$\sin(x)$										
$h(x)$										

A Polynomial Model

The $\sin(x)$ function has infinitely many x intercepts, so it follows that a polynomial of ‘infinite’ degree would be required; the square peg gets bigger!

Consider a polynomial defined as: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$, rather than modelling $\sin(x)$ using x intercepts, the model will be based on some simple calculus properties of $\sin(x)$.

First derivative: $\frac{d(\sin(x))}{dx} = 1$ when $x = 0$

Second derivative: $\frac{d^2(\sin(x))}{dx^2} = 0$ when $x = 0$

Third derivative: $\frac{d^3(\sin(x))}{dx^3} = -1$ when $x = 0$

Fourth derivative: $\frac{d^4(\sin(x))}{dx^4} = 0$ when $x = 0$

After the fourth derivative the sequence above cycles, so the fifth derivative becomes 1, the sixth becomes 0, the seventh becomes -1 and the eighth is 0 again. This property can be used to determine a polynomial model for $\sin(x)$ based on $p(x)$ rather than the intercept method with surprising results!

Question: 9.

Given $p(0) = 0$ determine the value of a_0 . [Step 1]

Question: 10.

Given $\frac{d(p(x))}{dx} = 1$ when $x = 0$, determine the value of a_1 . [Step 2 – First derivative]

Question: 11.

Given $\frac{d^2(p(x))}{dx^2} = 0$ when $x = 0$, determine the value of a_2 . [Step 3 – Second derivative]

Question: 12.

Given $\frac{d^3(p(x))}{dx^3} = -1$ when $x = 0$, determine the value of a_3 . [Step 4 – Third derivative]

Question: 13.

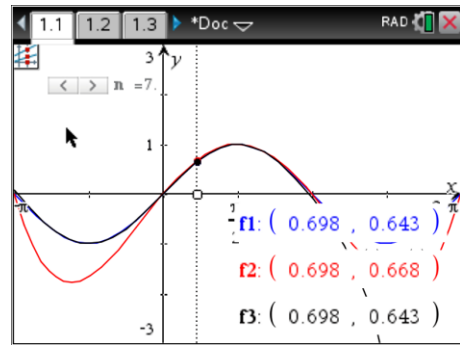
Given $\frac{d^4(p(x))}{dx^4} = 0$ when $x = 0$, determine the value of a_4 . [Step 5 – Fourth derivative]

Question: 14.

Use the process from the previous four questions to determine values for: a_5, a_6, a_7 and a_8 .

Define the function for $p(x)$ with the computed values for $a_0 \dots a_8$ and graph this function with $\sin(x)$ over the domain: $-\pi \leq x \leq 2\pi$.

A sample trace of the graphs is shown opposite. The graph of $f_1(x) = \sin(x)$, $f_2(x)$ is the quartic function and $f_3(x)$ is the new polynomial, notice the accuracy at this particular point?

**Question: 15.**

Define $p(x)$ as a polynomial of degree 7 and explain why, based on the previous calculations for a_8 , it is not necessary to use a polynomial of degree 8.

Question: 16.

Based on the patterns established so far, predict values for: a_9, a_{10}, a_{11} and a_{12} .

Question: 17.

Complete the table of values below, correct to FOUR decimal places. Comment on the accuracy of $p(x)$ to produce values for: $\sin(x)$.



Within the graph application, use the [menu] to change the **settings** for the graph application. Change the display digits to 4 decimal places.

x	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{4\pi}{9}$	$\frac{\pi}{2}$
$\sin(x)$										
$p(x)$										

Question: 18.

Given $\frac{d(\sin(x))}{dx} = \cos(x)$, determine a polynomial $q(x)$ for $\cos(x)$ and graph this function against $\cos(x)$. Comment on the accuracy of this polynomial.

Question: 19.

Define a rational function for $\tan(x)$ and determine the accuracy of this function.

