# Roots and Cobwebs
## Student Activity

Name ________________
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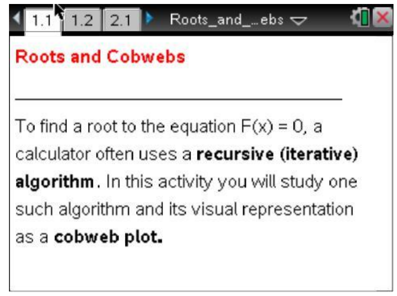
**Open the TI-Nspire document *Roots_and_Cobwebs.tns.***




To find a root to the equation $F(x)=0$, a calculator often uses a **recursive (iterative) algorithm.** In this activity, you will investigate one such algorithm, Fixed Point Iteration (FPI), and its visual representation as a **cobweb plot**.

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To use FPI to determine a sequence $\{x_n\}$ that converges to a root of F(x) = 0:

1) Rewrite $F(x)=0$ in the form $x=f(x)$ [there are often many choices for $f(x)$. ]

2) Choose an initial guess for a solution, $x_0$

3) Calculate $x_n=f(x_{n-1})$ *n* = 1, 2, .. until $\{x_n\}$ converges, i.e. $|x_n-x_{n-1}|$ is very small.

If $\{x_n\}$ does not converge, return to step 1, and make another choice for $f(x)$.

**Problem 1:** Estimate the root of $\sqrt{x}-x+1=0$ using FPI.

According to Step (1) above, we must rearrange $F(x)=\sqrt{x}-x+1=0$ into the form $x=f(x)$. One method to find a function $f(x)$ is: $\sqrt{x}-x+1=0 \rightarrow x=\sqrt{x}+1$ so that $f(x)=\sqrt{x}+1$. For Step (2), we choose $x_0=4.0$ as an initial guess for a root.

**Move to page 1.2.**

Press ctrl ▶ and ctrl ◀ to navigate through the lesson.

For Step (3), we can find the values $x_i, i=1,2,3,...$ using a spreadsheet. This process is illustrated on Page 1.2 for the initial guess $x_0=4.0$, using the following procedure:

- Type a title for the first column (**root1**).
- Type 4.0 into cell A1, and press enter.
- In cell A2, type $=\sqrt{a1}+1$ [ $=f(a_1)$ ], and press enter.
- Arrow up (▲) to select cell A2.
- Select **MENU > Data > Fill**.
- Press the down arrow (▼) until you reach cell A20, and press enter to fill in the cells.

The column of values suggests that the sequence converges to 2.61803.

1. Using Scratchpad, verify that 2.61803 is a root of $\sqrt{x}-x+1=0$.

 

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2. a. Using the spreadsheet, find whether the FPI sequence determined by $f(x)=\sqrt{x}+1$, with each given number below as the initial guess, converges to the root 2.61803.

   i) 1.0 ii) 8.0 iii) 12.0

   b. For what values of $x_0$ do you think the resulting FPI sequence will converge to the root 2.61803?

Note: To change the initial term in the sequence, change the value of cell A1 to the initial guess. The remaining values in the column corresponding to this initial guess are automatically updated.

3. Let's try another choice for $f(x)$, namely $f(x)=(x-1)^2$. Show how $\sqrt{x}-x+1=0$ can be rewritten in the form $x=f(x)$ to determine this choice of $f(x)$.

4. Does the FPI sequence using this choice for $f(x)$ with initial guess $x_0=4.0$ seem to converge to the root 2.61803? Use Column B on the spreadsheet and the procedure above to populate the column and answer the question.

5. Does the FPI sequence determined by $f(x)=(x-1)^2$ converge to the root 2.61803 for any choice(s) of the initial guess $x_0$?

Hint: Try various choices for the initial guess, $x_0$, in the second column of the spreadsheet used in Question 4.

**Problem 2a**: This problem illustrates the ancient Babylonian algorithm for estimating a square root. Perhaps the first algorithm used to approximate $\sqrt{a}$ is the "Babylonian method", or "Heron's method", named after the first-century Greek mathematician Heron of Alexandria who gave the first explicit description of the method. Heron's method is, perhaps, the first case where FPI was used historically.

Here is an illustration of this method for approximating $\sqrt{5}$ :

We can rewrite $F(x) = x^2 - 5 = 0$ as $x = \frac{5}{x}$ → $2x = x + \frac{5}{x}$ → $x = \frac{\left(x + \frac{5}{x}\right)}{2}$, so that

$f(x) = \frac{\left(x + \frac{5}{x}\right)}{2}$ is one choice for $f(x)$ .

> The logic behind this choice for $f(x)$ is that if $x$ is a good guess for $\sqrt{5}$, then so is $\frac{5}{x}$ since $x \cdot \frac{5}{x} = 5.$ Their average $\frac{\left(x + \frac{5}{x}\right)}{2}$ , then, should be a better guess.

**Move to page 2.1.**

One method of implementing FPI is to find $\sqrt{5}$ through iteration on a spreadsheet as in Problem 1. This process is illustrated for the initial guess $x_0 = 4.0$ on Page 2.1 by using the following procedure:

- Type a title for the first column ( $sqr5$ ).
- Type 4.0 into cell A1, and press [enter].
- In cell A2, type: $=(a1 + 5/a1)/2$ [ $= f(a_1)$ ], and press [enter].
- Arrow up (▲) to select cell A2.
- Select **MENU > Data > Fill**.
- Press the down arrow (▼) until you reach cell A20, and press [enter] to fill the cells.

The sequence of values seems to approach 2.23607 very rapidly since the fifth and sixth iterations were both equal to 2.23607.

6. a. Using the spreadsheet, find the fifth number in the FPI sequence determined by $f(x) = \frac{\left(x + \frac{5}{x}\right)}{2}$ , using each given number below as the initial guess.

   i) 6.0   ii) 1.0   iii) -8.0   iv) 10.0

   b. In each case, does the sequence seem to converge to $\sqrt{5}$ ? If not, to what does it seem to converge?

Note: To change the initial term in the sequence, change the value of A1 to the initial guess. The remainder of the values corresponding to this initial guess are automatically updated.

**Move to page 3.1.**

**Problem 2b: Cobweb Sequence Plotter**

- Define the iterative function $f(x)$ on Page 3.2.
- On Page 3.3, the graphs of $y = f2(x) = f(x)$ and $y = f1(x) = x$ are shown.
- The $k$-clicker steps through the iterations of $f$ composed with itself.
- The graph shows a starting point $(x_0, f(x_0))$ and the intermediate "caroms" to the line $y = x$.
- The $x$-clicker can be used to change the initial value of $x$ (or, alternatively, you can just drag the point on the x-axis to change the initial value).

**Move to page 3.2.**

The function $f(x) = \dfrac{\left(x + \dfrac{5}{x}\right)}{2}$ is defined on page 3.2.

**Move to page 3.3.**

The initial value of $x_0 = 4.0$ has been selected with the x-clicker.

7. Using the k-clicker, cycle through the values from $k$ = 0 to $k$ = 7.
   a. Explain how the segments going back and forth between $y = f(x)$ and $y = x$ in the graph illustrate a sequence of values tending to an estimate of $\sqrt{5}$.

   b. How do the values in the lower left of the screen compare to those in the spreadsheet?

8. Use the $x$-clicker to select each of the values of $x_0$ listed below, and then use the $k$-clicker to cycle though the values $k$ = 0 to $k$ = 7.

   i) 6.0 ii) 1.0 iii) -8.0 iv) 10.0

   a. In each case, does the cobweb graph illustrate the calculation of an estimate of $\sqrt{5}$?

   b. If not, what does it illustrate, if anything?

Note**:** You need to reset the $k$-clicker to 0 before considering a new value of $x_0$.You might have to type the initial value $x_0$ into the $x$-clicker to cycle through integer values of $x$.

9. Leave the graph corresponding to $x_0 = 10$ displayed. By dragging the initial value along the x-axis, determine the values of the initial guess $x_0$ for which the FPI sequence seems to converge to $\sqrt{5}$ (i.e., the sequence of segments converges to the point of intersection of the graphs $y = f1(x) = x$ and $y = f2(x) = f(x)$ in the first quadrant).

   What happens for the other values of $x$ between -10 and 10 for which the sequence does not seem to converge to $\sqrt{5}$ ?

**Problem 3:** The equation $x^3 + x - 6 = 0$ has a root between 1 and 2. Estimate this root using FPI.

Consider these four choices for $f(x)$.

i) $\sqrt[3]{6-x}$ ii) $6-x^3$ iii) $\dfrac{6}{x^2+1}$ iv) $\sqrt{\dfrac{6-x}{x}}$

10. Show how $x^3 + x - 6 = 0$ can be rewritten in the form $x = f(x)$ for choices iii) and iv).

**Move back to page 2.1.**

As we have seen in Problem 1, not all choices of $f(x)$ lead to a sequence converging to a root and some of the sequences that do converge to a root do so very slowly.

11. For each of these four choices of $f(x)$, use the process below, starting with an initial guess $x_0$, to determine whether the FPI sequence of values converge to the root. If yes, what is the root? If not, what happens to the sequence?
    - Type a title (root1, for example) for the next column (Column B, for example).
    - Type your initial guess $x_0$ (2.0 for example) into cell B1 (for example), and press enter.
    - In cell B2, type: $= f(b1)$ (For $f(x) = \sqrt[3]{6-x}$, type: $=(6-b1)^\wedge(1/3)$ and press enter.
    - Arrow up (▲) to select cell B2.
    - Select **MENU > Data > Fill**.
    - Press the down arrow (▼) until you reach cell B20, and press enter to fill the cells.

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**Move back to pages 3.2 and 3.3.**

Verify your results for Question 11 by creating the cobweb plot for each of the choices of $f(x)$ by first typing $define\ f(x) =$ (new choice for $f(x)$) on Page 3.2 and then examining the cobweb plots on Page 3.3.

12. Describe how the cobweb plot for each choice of $f(x)$ supports your answer for Question 11.

13. For each choice of $f(x)$ for which its FPI sequence converges to the root, use the process from Question 9 of dragging the initial value along the x-axis to determine the range of possible values of $x_0$.

**Extension**

Find the 3 roots of the cubic polynomial $x^3 - 3x + 1 = 0$ using FPI. Determine a suitable choice of $f(x)$ for each root, and verify that the FPI sequence of values approaches the root using both the spreadsheet and the associated cobweb plot.

Hint: You will need a different function, $f(x)$, for each root.