Imaginary Explosions

Student Activity

7 8 9 10 **11** **12**

TI-Nspire™   Investigation   Student   50 min

Introduction

What happens when you square a number, does it get bigger or smaller? What happens if you square the result, then keep on squaring the result over and over again? It doesn't take long to see what happens with some numbers:

   5, 25, 625, 390625, 15587890625, 23283064365386962890625 …

The number gets really big within just a few iterations. What about if you start with a small number:

   0.2, 0.04, 0.0016, 0.00000256, 0.0000000000065536, 0.00000000000000000000004294967296

Within just a few iterations our small starting number became really small!

Numbers between -1 and 1 become very small, -1 and 1 both end up at 1 and numbers outside this region become very big. This all seems relatively simple, but what happens if we change our real numbers for complex?

Investigation – Part I

Question: 1.

Just like real numbers, some results will be obvious almost immediately; others may take a little time to see what happens after repeated squaring. Perform 3 to 4 iterations (by-hand) for each of the following and determine whether the number is getting bigger, smaller or 'other'.

a) $1+i$

b) $\dfrac{1}{2}+\dfrac{1}{2}i$

c) $\dfrac{1}{4}+\dfrac{\sqrt{3}}{4}i$

d) $1-\dfrac{1}{2}i$

e) $\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}i$

f) $\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i$

g) $\dfrac{5}{13}+\dfrac{12}{13}i$

h) $1+\dfrac{1}{2}i$

i) $\dfrac{1}{2}+i$

Start a new TI-nspire file and insert the following applications:
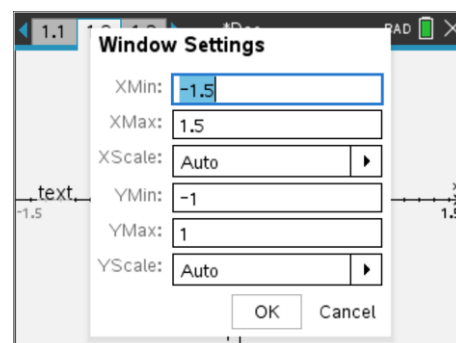
   Page 1.1: Calculator

   Page 1.2: Graphs

   Page 1.3: Lists & Spreadsheet

Navigate to the Graph application and use the menu to change the window settings.
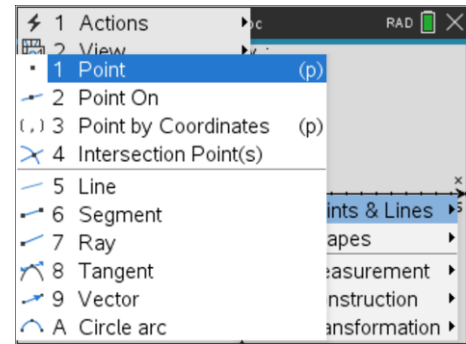
Window / Zoom > Window Settings

Adjust the window settings to match those shown opposite.

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This region will represent the Argand Plane. A point needs to be placed on the plane.

> **Geometry > Points & Lines > Point**

Do not put the point on the x or y axis.

Technology Tip!

On the TI-Nspire CX II press P on the keyboard rather than navigating through the menus, then select Point.

The coordinates of the point can be determined and automatically updated as the point is moved. Use either the contextual menu or:

> **Actions > Coordinates and Equations**

Select the point so that the point's coordinates are displayed on the screen. Press the ESC key to release the Coordinates command.

Try grabbing and moving the point around the screen to ensure the coordinates update automatically.
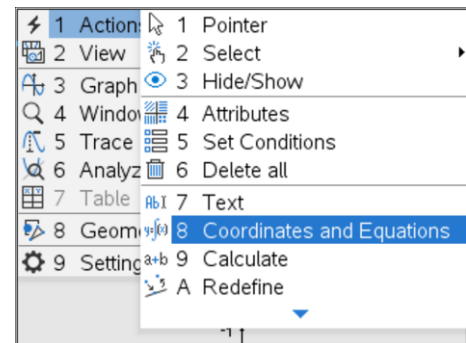
The coordinates of this point represent the real and imaginary components of a single imaginary number: $z = a + bi$

With the pointer over the x coordinate of the newly created point press:

> Ctrl + Menu   [Context menu]

Select:

> **Store**

Store the x coordinate as 'a', then repeat and store the y coordinate at 'b'
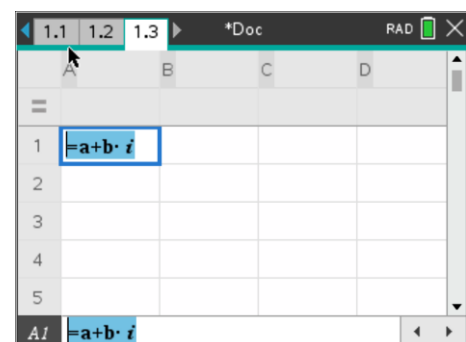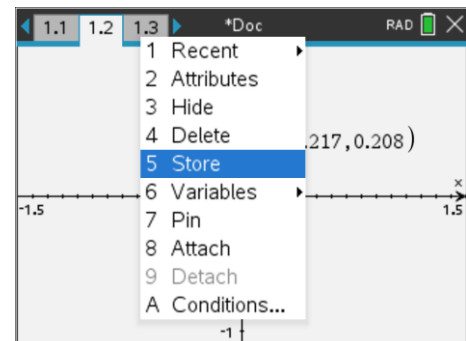
The coordinate points appear bold once stored.

Navigate to the Spreadsheet application.

In cell A1 type the formula:

> $= a + b \times i$

Make sure to use the complex number i rather than the keyboard i and start the expression with the equals sign.

The number in cell A1 needs to be squared and placed in cell A2.

To do this, type the following formula in cell A2:

= A1^2

Once the formula has been entered, navigate back to cell A2 and use the menu to select:

Data > Fill   [This enables us to fill the formula down]

Fill the data so that it occupies cells A1 through to A10.

To graph the sequence generated in column A, the real and imaginary components need to be isolated.

Navigate to the top of column B and type the heading:

"RD"   [Real data]

Now insert the heading "ID" at the top of column C.

In the formula bar for column B type the formula:

=real(data)

Now insert the formula "=imag(data)" in column C.

The data is ready to be plotted. Navigate back to the Graph application. Set the Graph type as a scatterplot. The real values need to be plotted on the x axis and the imaginary on the y.

The initial point can be dragged and moved, the corresponding sequence will respond accordingly.

The points can be joined and/or made smaller to focus on the patterns created by the sequence by changing the attributes.

To change the **attributes** place the pointer over a point on the scatterplot then press **Ctrl + Menu**.

Question: 2.

Drag the starting point: (a, b) around the Argand plane. Sequences fall into one of two categories: those that spiral towards 0 and those that seem to almost randomly spiral outwards. Record at least 10 points noting which ones spiral in and which ones spiral out. Determine the magnitude of each starting point.

Investigation – Part II

The recursive process will now be extended to include a constant. Each time before the number is squared a constant will be first added to the new number. This process can be summarised as:

$$z_{n+1} = z_n + c$$

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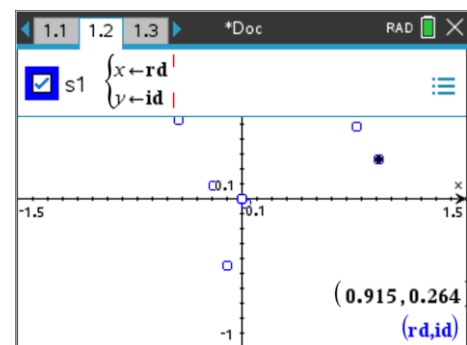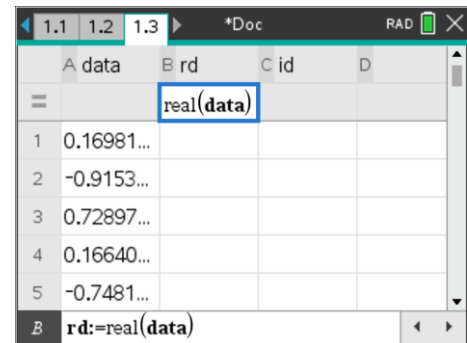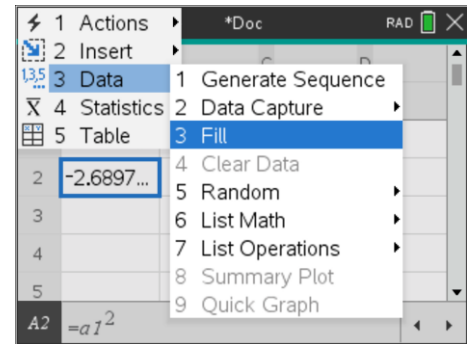
Navigate to the Graph application.

- Insert another point
- Generate the coordinates of the point
- Store the x coordinate as cx
- Store the y coordinate as cy
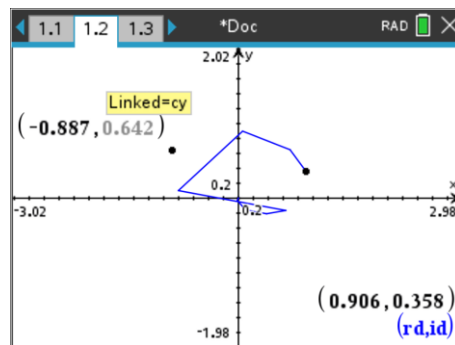
In the spreadsheet application edit the formula in cell A2 to include the constant:

$$= a1^2 + \mathbf{cx} + \mathbf{cy} \times \boldsymbol{i}$$

Fill the new formula in cell A2 down to cell A15.

Note that since columns B (rd) and C (id) use column formulas based on the data in column A, these columns will automatically extend to encapsulate the new calculations added to cells A11: A15.

Return to the Graph application and drag the original point representing $z = a + bi$ so that it is as close to zero (origin) as possible. Now drag the new point representing c, the constant, around the screen and observe some of the patterns that occur.

Experiment with switching the scatter plot between connected (shown) and disconnected.
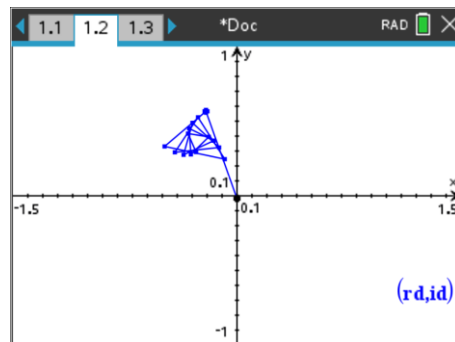
Question: 3.

Identify some of the patterns that form with this new recursive relationship. (Capture some screen images)

Question: 4.

Has the spiral in/out region changed?

Investigation – Part III

This new recursive process $z_{n+1} = z_n + c$ creates many interesting patterns. This section explores the 'boundary' where the sequence tends to spiral out, exploding the magnitude of the complex numbers and those that spiral in to some other value.

Insert a Notes application into the document. The Notes application will be used to monitor the magnitude of the last number generated by the sequence, the value in cell A15.

In the Notes application insert a Maths box. Ctrl + M

Type the formula:

d :=abs(data[15])

The variable: " d " now holds the magnitude of the last calculated term in the sequence. This value will be updated as the constant on the Graph application is moved around the screen.

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Navigate to the Graph application and place the mouse over the point that represents the constant c. (c_x, c_y).

> Press: **Ctrl + Menu**

This activates the contextual menu. Select the Conditions option.

The point can change appearance based on a set of specified conditions. In this case the point will change colour if it appears that it is about to spiral out and the magnitude effectively exploding.

Enter the condition displayed for the Line colour:

> when(d < 5, 2, 7)

The limit of '5' is arbitrary and can be replaced with a larger number.

Question: 5.

> With z located as close to the origin as possible, drag the constant around the screen and determine what happens to the colour of point C when the sequence is spiralling outwards?

We are interested in the region and its corresponding boundary where the constant causes the sequence to explode in magnitude. It is difficult to visualise this region as it appears somewhat irregular. To help with this process a series of points will be captured.

Return to the spreadsheet and label two more columns: XP and YP.

Navigate to the formula line and use the menu to activate manual data capture:

> **Menu > Data > Data Capture > Manual**

In column XP capture the real value (CX) of the current constant.

In column YP, capture the imaginary value (CY) of the current constant.

Values will only be added to these lists when they are manually generated on the Graph application.

Return the Graph application and generate a scatterplot of (XP, YP).

No values will be plotted at first since these two lists are empty.

Grab the constant (CX, CY) and move it to a point where the sequence suddenly changes from spiralling in to spiralling out. (Point changes colour)

Press Ctrl + '.'   [Note that 'capture' is written above the decimal point]

A point is recorded. Move to another location where the constant just changes the sequence from spiral in to spiral out. Capture this new point by again pressing Ctrl + '.' . Repeat this process until a boundary starts to form.

Question: 6.

> Focus your search in one quadrant. The boundary region between 'exploding' and stabilising is somewhat irregular so it will take a while to capture sufficient points. Experiment with moving in/out of your region as it develops.
> Use a screen capture to illustrate the region.

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