

# Vector Graphics

## Student Activity

7 8 9 10 11 12



TI-84PlusCE™



Investigation



Student



50 min

## Introduction

In 1637 René Descartes published a revolutionary book *La Géométrie* where he brought together two areas of mathematics; Geometry and Algebra. The “Cartesian” plane, named after René Descartes, is an integral component of every high school mathematics course. In this activity you will see how vectors can be incorporated into this two dimensional space. Using the same techniques, you can easily extend this approach into three dimensions.

Vectors can be expressed as a direction and magnitude (Polar) or as mutually perpendicular projections (Components), the latter form is explored in this activity. The directions are referred to as  $i$  ( $x$  direction) and  $j$  ( $y$  direction), for three dimensions a  $k$  ( $z$  direction) is also included. For some relatively straight forward problems it is quicker and more appropriate to complete calculations ‘by-hand’, however, as the calculations are repeated and gradually increase in complexity, having a template becomes an extremely useful tool. This activity sets up a template that can be used for other investigations including problems in three dimensions.

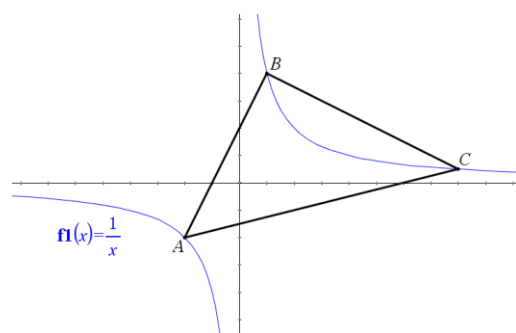
## Problem to be explored

Consider three points A, B and C on the function:  $y = \frac{1}{x}$ .

These three points can be joined to create a triangle:  $\triangle ABC$ .

This triangle has some interesting features that will be explored through the use of vectors.

To begin this investigation, point A = ( -1, -1 ), B = (  $\frac{1}{2}$ , 2 ) and C = ( 4,  $\frac{1}{4}$  ).



## Vectors as Matrices

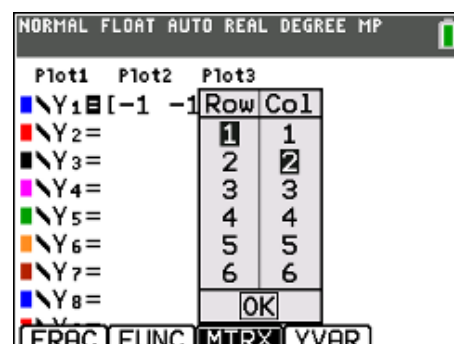
A vector in **component** form can be used to reference point A:

$$OA = \underline{a} = -\underline{i} - \underline{j}$$

One way to represent a vector on the calculator is to use a matrix. This vector can be stored in Matrix A; however, for this investigation it will be stored directly into  $Y_1$ .

Press  $\boxed{y=}$  to access the editor, followed by  $\boxed{\alpha}$  and  $\boxed{[r3]}$  for the Matrix insertion shortcut.

Use this menu to insert a  $1 \times 2$  matrix to represent the vector OA.



Use the above process to create vectors OB and OC in  $Y_2$  and  $Y_3$  respectively.

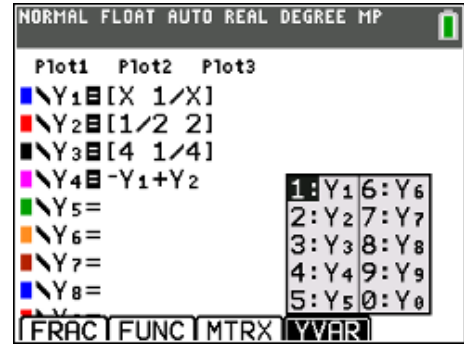
Vector AB is equal to:  $AO + OB$  or  $-OA + OB$

Vectors AB, BC and CA will be stored in  $Y_4$ ,  $Y_5$  and  $Y_6$  respectively.

Within the Y = editor, navigate to  $Y_4$  and press  $\boxed{\text{vars}}$ , arrow across to Y-VARS and select Function followed by  $Y_1$ , alternatively press  $\boxed{\text{alpha}}$   $\boxed{[f4]}$  and select the corresponding function.

To calculate  $-OA + OB$ , enter the expression:  $-Y_1 + Y_2$

$Y_4$  now contains vector AB.



**Question 1**

On the calculator home screen use the  $\boxed{\text{vars}}$  key and function menu to recall  $Y_4$  and hence determine the matrix that represents vector AB. Explain your answer in terms of point A and B.

**Question 2**

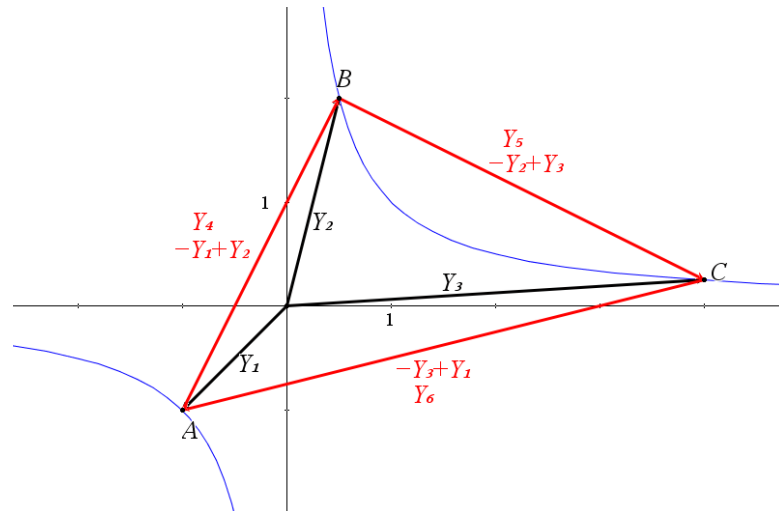
Determine expressions for vectors BC and CA in terms of OA ( $Y_1$ ), OB ( $Y_2$ ) and OC ( $Y_3$ ) as applicable. Store these expressions in  $Y_5$  and  $Y_6$  respectively. Write down the matrix and equivalent vector expressions for BC and CA.

**Question 3**

On the calculator home screen calculate:  $Y_4 + Y_5 + Y_6$  and explain the result.

**Question 4**

Draw a diagram of the graph and all six vectors. (Label each vector as  $Y_1, Y_2 \dots Y_6$ )



## Magnitude of a Vector

The magnitude of vector is calculated using Pythagoras's theorem, squaring each component, adding these squared values together and then calculating the square-root:

The magnitude of a vector can be computed by:

$$|\underline{a}| = \sqrt{x^2 + y^2} \quad \text{where } \underline{a} = x\underline{i} + y\underline{j}$$

We can see how this works with matrices:

$$\underline{a} = [x \quad y]$$

To multiply matrix A by itself requires the matrix to be transposed so the rows and columns align:

$$(\underline{a})^T = \begin{bmatrix} x \\ y \end{bmatrix}$$

The product is therefore:

$$\underline{a} \times \underline{a}^T = [x \quad y] \begin{bmatrix} x \\ y \end{bmatrix} = [x^2 + y^2]$$

The determinant of this 1 x 1 matrix is simply  $x^2 + y^2$  which then allows the square-root of this value to be computed.

**Note:** The determinant and transpose commands are located in the matrix menu:  $\boxed{2\text{nd}}$  [Matrix] > MATH

```
NORMAL FLOAT AUTO REAL DEGREE MP
NAMES MATH EDIT
1:det(
2:T
3:dim(
4:Fill(
5:identity(
6:randM(
7:augment(
8:Matr▶list(
9↓List▶matr(
```

```
NORMAL FLOAT AUTO REAL DEGREE MP
√det(Y4*Y4T)
..... 3.535533906
```

## Mathematics

### Tip!



In “vector world”, the magnitude of a vector can also be computed using the dot product:

$$|\underline{a}| = \sqrt{\underline{a} \cdot \underline{a}}$$

## Question 5

Determine the magnitude of vectors AB, BC and CA and hence the perimeter of triangle ABC.

Suppose point A (-1, -1) is moved to a new location on the curve  $y = \frac{1}{x}$ , the perimeter of the triangle would obviously change.

Use the Y = editor to set the coordinates of point A as:  $\left(x, \frac{1}{x}\right)$

Using a variable in the matrix can only be achieved whilst the matrix is associated with a Y= equation. This approach allows the matrix to be used in a ‘solve’ command.

```
NORMAL FLOAT AUTO REAL DEGREE MP
Plot1 Plot2 Plot3
Y1 [X 1/X]
Y2 [1/2 2]
Y3 [4 1/4]
Y4 -Y1+Y2
Y5 -Y2+Y3
Y6 -Y3+Y1
Y7 =
Y8 =
Y9 =
```

Suppose point A is moved such that the perimeter is now equal to 17. This means that the sum of the side lengths, vector magnitudes, will be equal to 17.

$$|AB| + |BC| + |CA| = 17$$

The numerical equation solver can be accessed by pressing:

$\boxed{\text{math}}$  > Option C (Numeric solver)

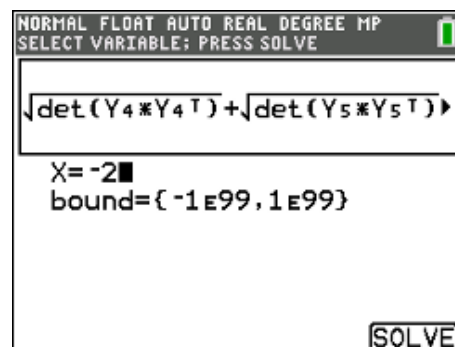
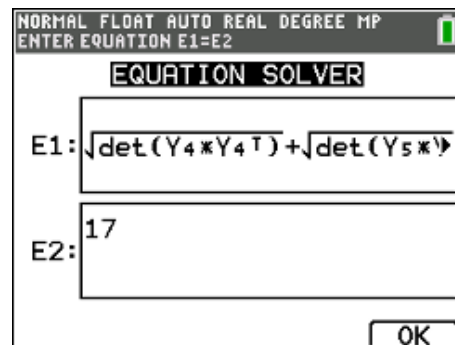
Enter expressions for the magnitude of each vector for E1 (Equation 1) and the perimeter for E2 (Equation 2) as shown.

In this menu the  $\boxed{\text{GRAPH}}$  key becomes a 'soft' key for the "OK" option.

Select "OK", edit the approximate starting value for X to direct the calculator where a solution may be located.

Select "SOLVE" ... the calculator will search for an answer, as indicated by the 'spinning wheel' adjacent to the Battery symbol.

This process may be hastened by reducing the default boundary conditions:  $\{-1 \times 10^{99}, 1 \times 10^{99}\}$



### Calculator Tip!



For large and repetitive calculations it can be more efficient to store an expression as a 'string'. Place quotation marks around the expression and use  $\boxed{\text{sto} \rightarrow}$  and  $\boxed{\text{vars}}$  > String to store.

The mathematical expression can then be 'recalled' by using the 'recall' operation (RCL) located above  $\boxed{\text{sto} \rightarrow}$  and selecting the corresponding String variable from the  $\boxed{\text{vars}}$  menu.

### Question 6

What location(s) are possible for point A to generate a perimeter of 17?

### Dot Product of a Vector

The dot product of two vectors is computed as follows:

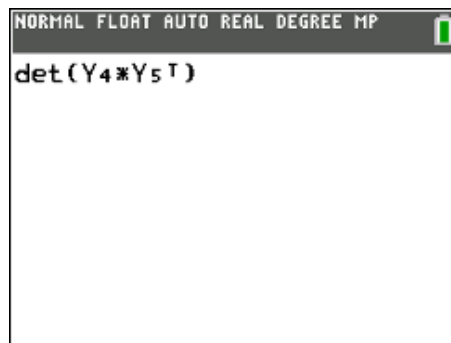
$$\underline{a} \cdot \underline{b} = x_a x_b + y_a y_b + z_a z_b \quad \text{where: } \underline{a} = x_a \underline{i} + y_a \underline{j} + z_a \underline{k} \quad \text{and} \quad \underline{b} = x_b \underline{i} + y_b \underline{j} + z_b \underline{k}$$

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos(\theta) \quad \text{where } \theta \text{ is the angle between the two vectors.}$$

The dot product of two vectors can be used to compute the angle between the vectors, in this section however we will use the fact that if the dot product is equal to zero then the vectors are at right angles. ( $\cos(90^\circ) = 0$ )

It is easy to calculate the dot product of the vectors stored in  $Y_4$ ,  $Y_5$  and  $Y_6$ , the vectors that define the sides of the triangle.

$$\det(Y_4 * Y_5^T)$$



### Question 7

Change the location of point A to:  $(-1, -1)$  then calculate the dot product of vectors AB and BC ( $Y_4$  and  $Y_5$ ). What does the result say about the angle between these two sides?

### Question 8

Using your answer to question 6 to help determine the area of the triangle ABC where A is located at  $(-1, -1)$ .

Suppose the vertices of the triangle are now:

$$A = (-1, -1)$$

$$B = \left(\frac{1}{3}, 3\right)$$

$$C = \left(c, \frac{1}{c}\right)$$

Point C is to be moved such that a right angled triangle is to be formed once again.

Edit  $Y_3$  to include 'c'. Note that  $c \neq 0$ .

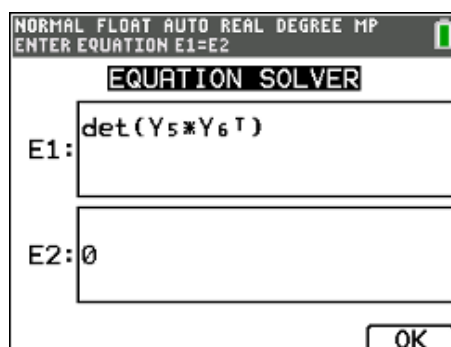
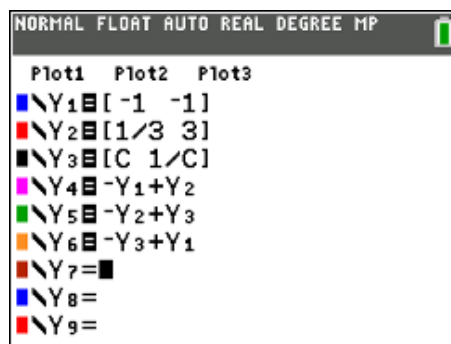
The calculator's numeric solver can now be used to locate point C.

We are looking for vectors BC and CA to be perpendicular, therefore

$$BC \cdot CA = 0 \quad \text{or} \quad \det(Y_5 * Y_6^T) = 0$$

This can be entered into the numeric solver: Math > Numeric Solver

E1 represents Expression 1, similarly E2 represents Expression 2.

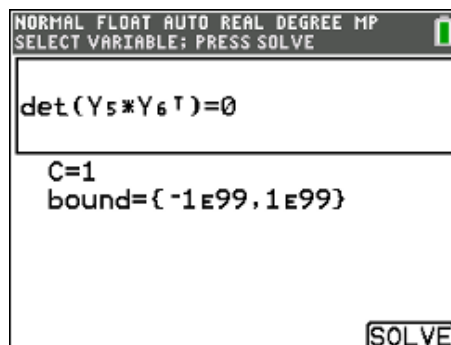


The equation to be solved is:

$$\text{Expression 1} = \text{Expression 2} \quad (\text{as shown})$$

Note that the value entered in C is the current value stored in C. If this current value is zero, then it will need to be changed before attempting to solve the problem since  $C \neq 0$ .

If the process is slow, the boundary conditions can be narrowed manually to expedite the computations or if specific values are sought.



### Question 9

What is the value of C for which ABC is once again a right angled triangle?

Note: This can be expressed as an exact value.

**Question 10**

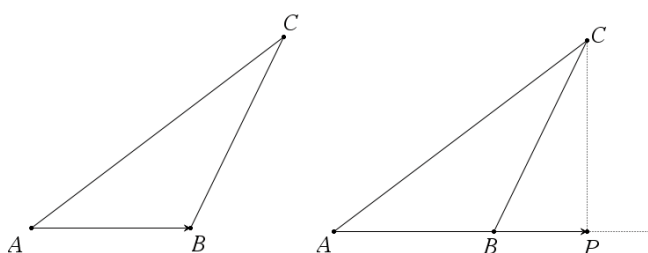
Verify that the value of C forms a right angled triangle.

**Calculator****Tip!**

After using the Numeric Solver command, the computed value is automatically stored in the corresponding variable. This means that the vectors BC ( $Y_5$ ) and CA ( $Y_6$ ) will now be equal to the vectors that form a right angled triangle.

**Vector multiplied by a Scalar**

When a vector is multiplied by a scalar, the direction<sup>1</sup> of the vector remains the same but the length will change if the scalar quantity is not equal to 1. A simple way to explore this is through a geometric example.

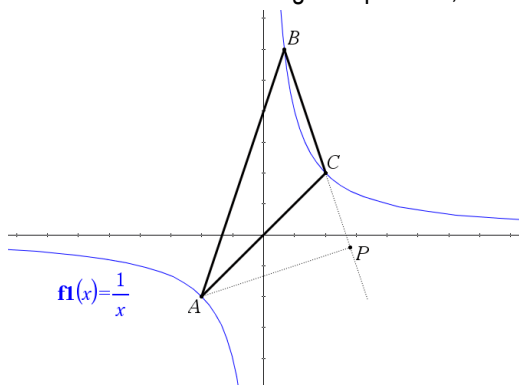


Consider the triangles shown here. When trying to calculate the height of  $\triangle ABC$  we can extend the base to point P such that  $\angle APC = 90^\circ$ .

Vector AB can be multiplied by a scalar  $n$  such that vector  $AP = nAB$ .

Vector AP has the same direction as AB but has a different length. **Note:** PC is the altitude of  $\triangle ABC$ .

For this section of the investigation points A, B and C are defined as:



$$A = (-1, -1) \quad B = \left(\frac{1}{3}, 3\right) \quad C = (1, 1)$$

Notice that BC has been extended such that  $BP = nBC$ .

P is located such that PA is perpendicular to BP.

This means that  $AP \cdot BP = 0$

**Note:** AP is the altitude of  $\triangle ABC$ .

**Question 11**

Write a vector equation for AP in terms of BC and AB.

**Question 12**

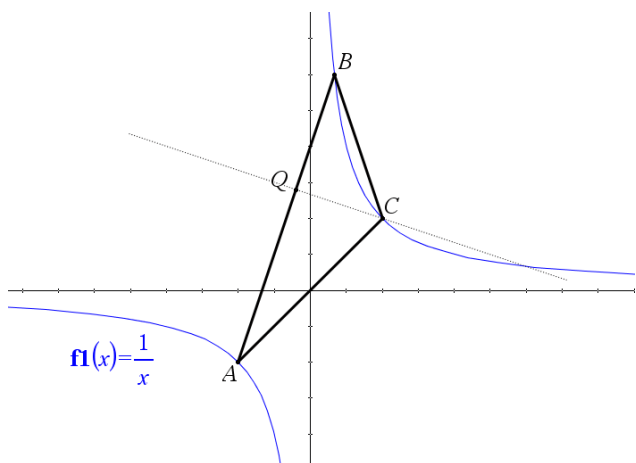
Determine the value of  $n$  such that  $AP \cdot BP = 0$

Note: The numerical solver on the calculator can be used to determine the value of  $n$ .

**Question 13**

Determine the vector OP and hence the coordinates of point P.

<sup>1</sup> Direction – The direction will reverse if the scalar is a negative quantity.



$$A = (-1, -1)$$

$$B = \left(\frac{1}{3}, 3\right)$$

$$C = (1, 1)$$

Point Q is such that AQ is perpendicular to CQ.

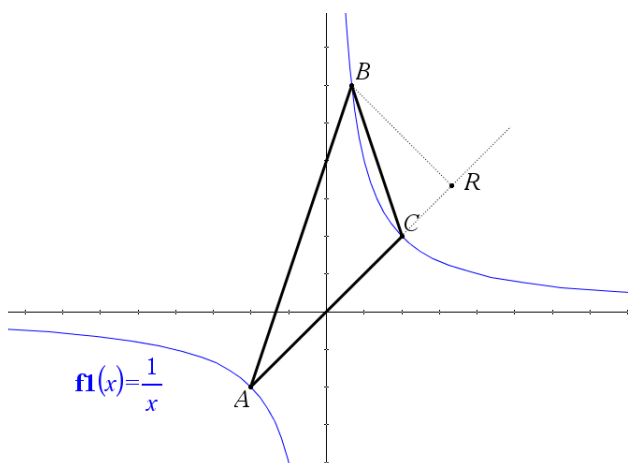
**Note:** CQ is another altitude of  $\triangle ABC$ .

#### Question 14

Write a vector equation for CQ in terms of CA and AB.

#### Question 15

Determine the value of  $m$  such that  $AQ \cdot CQ = 0$  and hence the vector OQ.



$$A = (-1, -1)$$

$$B = \left(\frac{1}{3}, 3\right)$$

$$C = (1, 1)$$

Point R is such that AR is perpendicular to BR.

**Note:** BR is the third altitude of  $\triangle ABC$ .

#### Question 16

Write a vector equation for BR in terms of AB and CA.

#### Question 17

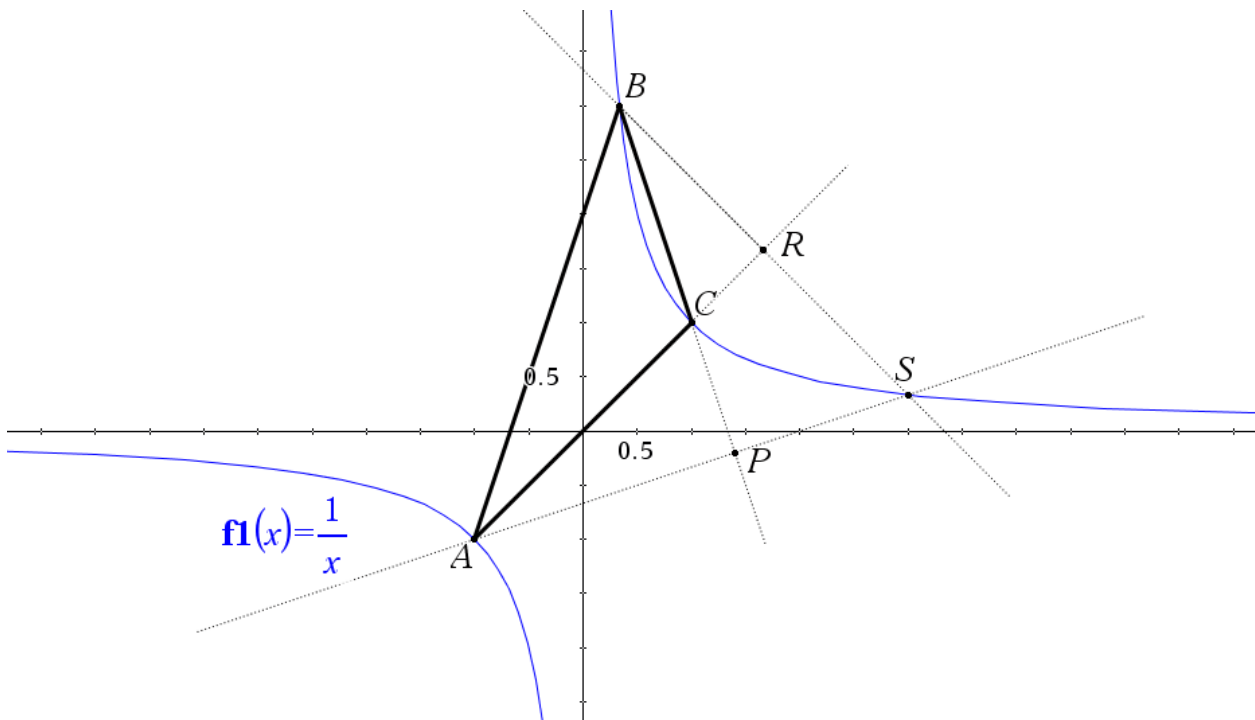
Determine the value of  $s$  such that  $sAC \cdot RB = 0$  and hence the vector OR.

#### Question 18

Consider the values of  $m$ ,  $n$ , and  $s$  established in the previous questions and their respective locations. By considering the values only, how is it possible to tell if the altitude (perpendicular) sits on the vector?

#### Extension: Points of Intersection Investigation

It is possible to determine where each of the altitudes intersects one another. The diagram below shows the point of intersection between altitudes BR and AP at point S.



**Important knowledge:**

- $BS$  is a scalar multiple of  $BR$ .
- $AS$  is a scalar multiple of  $AP$ .
- $OS = OB + BS$  and  $OS = OA + AS$ .

The above information can be used to determine the location of  $OS$ . Determine the location of  $S$  and explore the location of the point(s) of intersection for the altitude passing through  $C$  with those passing through  $A$  and  $B$ .

Vary the locations of points  $A$ ,  $B$  and  $C$  and summarise the location of point  $S$  and the general vector defining  $OS$ .