Math Objectives

- Students will be able to visualize the solid generated by revolving the region bounded between two function graphs and the vertical lines $x = a$ and $x = b$ about the x-axis.
- Students will be able to calculate the area of a cross section of a solid generated by rotating the region bounded between two function graphs and the vertical lines $x = a$ and $x = b$ about the x-axis.
- Students will be able to use the area of a cross section of a solid generated by rotating a region bounded by two function graphs about the x-axis to find the exact volume of the solid.
- Students will look for and make use of structure. (CCSS Mathematical Practice)
- Students will model with mathematics. (CCSS Mathematical Practice)

Vocabulary

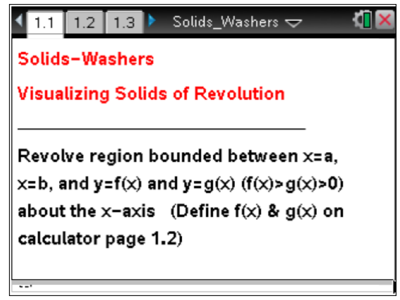
- revolve about the x-axis
- solid
- washer

About the Lesson

- The intent of this lesson is to provide students with visual representations of solids of revolution.
- As a result, students will:
  - Observe that the solid generated by revolving the region bounded between two functions about the x-axis has cross sections shaped like washers centered on the x-axis, the radii of which correspond to the values of the upper function and the lower function at that point.
  - Observe that the volume of a solid of revolution can be estimated using a sum of volumes of washer slices.
  - Observe that the exact volume can be found by integration.

TI-Nspire™ Navigator™ System

- Use Quick Poll to assess student understanding.
- Use Screen Capture to demonstrate different methods of estimating area under a curve.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- In Graphs, you can hide the function entry line by pressing ctrl G.

Lesson Materials:

Student Activity
Visualizing_Solids_of_Revolution_Washers_Student.pdf
Visualizing_Solids_of_Revolution_Washers_Student.doc

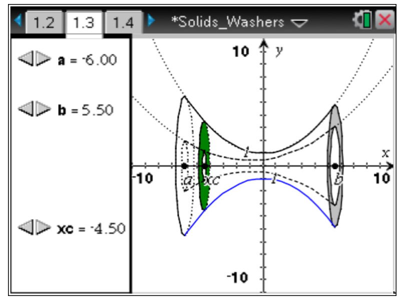
TI-Nspire document
Visulaizing_Solids_of_Revolution_Washers.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

> **Tech Tip:** While it is also possible to drag the points to change the values of *a*, *b*, and *xc* along the *x*-axis, the slider arrows provide much better control for the students, especially on the handheld units.

Move to page 1.3.



1. The graphs of the functions $\mathbf{f}(x) = \frac{x^2}{8} + 1$ and $g(x) = \frac{x^2}{16} + \frac{x}{8} + \frac{1}{2}$ are shown on page 1.3. The region between the two curves has been bound between $x = a$ and $x = b$ and rotated about the x-axis.

   a. Which function is the upper function? Which is the lower? How do you know?

   <u>Answer:</u> The top function is $\mathbf{f}(x) = \frac{x^2}{8} + 1$ and the bottom function is $\mathbf{g}(x) = \frac{x^2}{16} + \frac{x}{8} + \frac{1}{2}$.
   Explanations may vary. For example, the y-intercept of $\mathbf{f}(x)$ is 1 and the y-intercept of $\mathbf{g}(x)$ is $\frac{1}{2}$, so students may be able to identify the graphs in this way, and then observe that, at least in the range over which they are looking, the graph of **f** remains above the graph of **g**.

   b. What does the solid look like? Describe its shape.

   <u>Answer:</u> The solid resembles a tube that is smallest at $x = 0$ and grows larger on all sides as x increases and decreases. It is essential to note that the solid has an open hole through its middle. (Some students may describe it as a vase with thick sides and a hole all the way through.)

   c. Imagine you sliced through the solid region, slicing perpendicular to the x-axis. What would the cross section look like? Explain.

   <u>Answer:</u> The cross section would be circular, with an open middle, resembling a washer or a CD (the geometric term is *annulus*).

d. Does the shape of the cross section depend on the location where you make your slice? Explain.

Answer: The cross section will be a washer wherever the cross section is taken. However, the size of the washer will change depending on the location of the cross section.

e. Move point *xc* along the *x*-axis using the slider. Does this support or contradict your prediction from part 1d? Explain.

Answer: This supports the prediction. The shape seems to stay consistent as the point is dragged (but the size of the outer edge and the hole varies).

2. Move point *xc* until it is equal to 3. Suppose you slice through the solid at that point.

a. What is the shape of the cross section at that location? Explain.

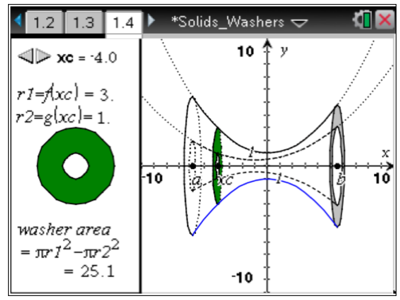
Answer: It is still a washer because the cross section is bound by two concentric circles.

b. What is the area of the cross section when *xc* = 3? How did you determine this?

Answer: The cross section at *xc* = 3 is a circular region with a smaller circular region removed. The radii of the two circular regions are determined by the values of the functions there. The outer radius is the value of **f**(*x*) at 3, $r_1 = \mathbf{f}(3) = \frac{3^2}{8} + 1 = \frac{17}{8}$. The inner radius is the value of $\mathbf{g}(x)$ at 3, $r_2 = \mathbf{g}(3) = \frac{3^2}{16} + \frac{3}{8} + \frac{1}{2} = \frac{23}{16}$. Thus the area of that circular cross section is $\pi(r_1)^2 - \pi(r_2)^2 = \pi \cdot \left(\frac{17}{8}\right)^2 - \pi \cdot \left(\frac{23}{16}\right)^2 = \frac{627}{256}\pi$.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 1 at the end of this lesson.

Move to page 1.4.



3. Move point *xc* until it is equal to 3. Note that the cross section of the solid at that point is pictured on the left.
   a. How is the area of the washer at *xc*, shown on the left, calculated?

   Answer: The radii are the values of the function at *xc* = 3. The area is found by subtracting the area of the smaller region from the area of the larger circle.

   b. How does this compare to the area you found in question 2b? Explain.

   Answer: The answer is $\frac{627}{256}\pi$ rounded to the nearest hundredth.

   c. How could you express the area of a cross section taken at any point *x* between *a* and *b?* Explain.

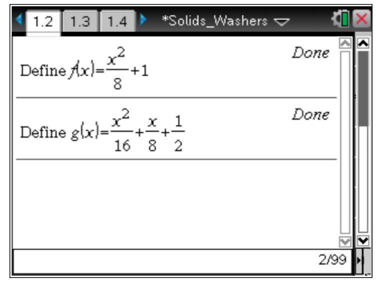
   Answer: At any *x*-value, the cross section will still be a washer, so the area of the cross section will be $\pi(r_1)^2 - \pi(r_2)^2$, where r_1 is the radius of the outer circle and r_2 is the radius of the inner circle. But the radii are determined by the values of the function at that *x* value. Thus the area will be $\pi\left(\frac{x^2}{8}+1\right)^2 - \pi\left(\frac{x^2}{16}+\frac{x}{8}+\frac{1}{2}\right)^2$.

TI-Nspire Navigator Opportunity: *Quick Poll (Multiple Choice* or *Open Response)*
See Note 2 at the end of this lesson.

   d. Do you think you will be able to find the area of a cross section in the same way for any region bounded between two functions that is rotated about the *x*-axis? Explain.

   Answer: As long as you have one function that is always above the other function, and both functions are above (or below) the *x*-axis, you should be able to do this in the same way.

> **Teacher Tip:** Note that **f** may be greater than **g** on part of the interval and less than **g** on another part of the interval. In this case, the area of the cross sections can still be determined using washers, but the “upper” and “lower” functions will depend on the location of the cross section.



Move back to page 1.2.

4. Click on the screen and type in **Define f(*x*) =** and a positive function of your choice. Then type in **Define g(*x*) =** and a second positive function of your choice.
   a. What do you think the solid generated by rotating your function between *a* = –3 and *b* = 6 will look like?

   Answer: Student answers may vary. They should note that it will still have washer cross sections, but different functions will have different shapes.

   > **Teacher Tip:** You may want to give your students a set of functions to choose from in order to ensure that they have functions whose solids will be easy to see in a standard window, and that they have two functions with the property that one is always greater than the other. Alternatively, you may wish to allow students to select any function, creating opportunities to troubleshoot graphing windows, to address issues of discontinuity, etc.

   b. What do you think a cross section of the solid generated by rotating the region bound by your functions about the *x*-axis will look like? Explain.

   Answer: If the functions are continuous everywhere in the interval over which the region is bound, the cross sections will be washers.

   c. What do you think the area of a cross section taken at *xc* = 1 will be? Explain.

   Answer: The area will be $\pi(\mathbf{f}(1))^2-\pi(\mathbf{g}(1))^2$ or $\pi(\mathbf{g}(1))^2-\pi(\mathbf{f}(1))^2$ where **f**(*x*) and **g**(*x*) are the student's functions. The choice will be based on which of the student's functions are greater at *x* = 1.

Move to page 1.4.

You may have to adjust your window to get a good view of your function rotated about the *x*-axis.

> **TI-Nspire Navigator Opportunity:** ***Screen Capture***
> **See Note 3 at the end of this lesson.**

5. a. Were your predictions from question 4 correct? Explain.

   Answer: Answers may vary.

   b. How could you express the area of a cross section taken at any *x*-value between *a* and *b*?

   Answer: The area will be $\pi(\mathbf{f}(x))^2 - \pi(\mathbf{g}(x))^2$ where **f**(*x*) and **g**(*x*) are the student's functions.

6. Set *a* = –3 and *b* = 6. Place *xc* anywhere between *a* and *b*. Suppose that you take a thin slice of the solid at *xc,* shaped like a washer. Call the thickness of the washer Δx.

   a. What is the smallest value *xc* can have? The largest?

   Answer: Smallest: *xc* = –3; largest: *xc* = 6.

   b. What is the volume of the washer you have sliced from the solid? Explain.

   Answer: The volume is $\pi(\mathbf{f}(x))^2 - \pi(\mathbf{g}(x))^2\,\Delta x$, provided **f** > **g** between *x* = *a* and *x* = *b*.

   c. Imagine that you slice the whole solid into washers of thickness Δx. How could you estimate the total volume of the solid using these washers? Explain. (Hint: Think back to Riemann sums.)

   Answer: $\sum \pi(\mathbf{f}(x))^2 - \pi(\mathbf{g}(x))^2\,\Delta x$, over all the ***xc***'s upon which the interval has been divided (again, provided **f** > **g** on the interval in question).

   d. How could you find the exact volume of your solid? Explain. (Hint: Think back to moving from Riemann sums to exact area under a curve.)

   Answer: $\int_{-3}^{6} (\pi(\mathbf{f}(x))^2 - \pi(\mathbf{g}(x))^2)dx$. Analogous to Riemann sums, you take the limit of the sum in part 6c, where Δx is going to 0. This is the integral over the values *xc* can take, –3 to 6.

   e. Will this work for any pair of functions? Explain.

   Answer: Yes, it will work for any pair of functions that are continuous over the interval. Cross sections will always be washers (as long as both functions are above the *x*-axis and **f** > **g** on the interval). You can always build the solid out of these washers. Letting the thickness of the slices go to 0, you get the exact volume by calculating the integral $\int_a^b (\pi(\mathbf{f}(x))^2 - \pi(\mathbf{g}(x))^2)dx$.

Wrap Up

Upon completion of this discussion, the teacher should ensure that students understand:

- That the cross sections of a solid of revolution of a region bounded by continuous functions, one of which is always greater than the other, on a bounded interval about the *x*-axis are washers.
- The volume of such solids can be estimated by summing cross sectional washers.
- The volumes of such solids can be found by integrating the cross sectional areas.

Assessment

Have students find the volumes of other students' solids.

Ask students when this method will not work. What if you revolved about a different axis? What if one of the functions was not continuous? What if the order of the functions switched at one or more points in the interval? What if you took slices parallel to the *x*-axis?

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Note 1

Question 2b, *Quick Poll:* Use a *Quick Poll* to find students' determinations of the area. Opportunities may arise to discuss why using the formula for area of a circle makes sense.

Note 2

Question 3c, *Quick Poll:* Use a *Quick Poll* to determine if students think cross sections will always be washers and the radii will always be the values of the functions there. This could be used as an entry point for a whole class discussion.

Note 3

Question 5, *Screen Capture:* Use a *Screen Capture* to show the solids generated by revolving regions bounded by different functions about the *x*-axis. This could create opportunities to discuss what cross sections will look like and if they will ever be different.