Summing Up Geometric Series

About the Lesson

In this activity, students will explore infinite geometric series and the partial sums of geometric series. The students will determine the limits of these sequences and series using tables and graphs. As a result, students will:

- Derive and apply a formula for the sum of an infinite convergent geometric series.
- Use the $\sum$ template to verify the formula for the sum of an infinite series in specific cases.
- Prove and apply the ratio (of consecutive terms) test to prove a series convergent or divergent.
- Prove that a necessary condition that a geometric series converges is that $|r| < 1$ where r is the common ratio.

Vocabulary

- common ratio
- infinite series
- partial sum
- convergence
- divergence

Teacher Preparation and Notes

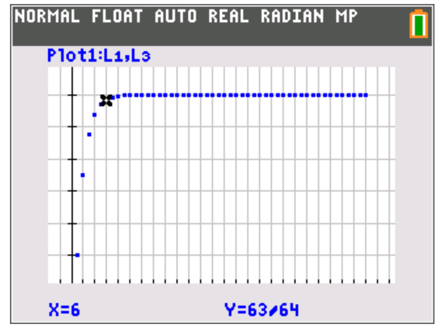
- Students should already be familiar with sequences, partial sums, and the definition of convergent and divergent series.
- This activity is designed to be teacher-led. You may use the following pages to present the material to the class and encourage discussion. Students will follow along with their calculators. Although the majority of the ideas and concepts are only presented in this document, be sure to cover all the material necessary for students' comprehension.
- Before beginning the activity, students should press 2nd [mem] and select **4:ClrAllLists** to clear all data from their lists.

Activity Materials

- Compatible TI Technologies:

  TI-84 Plus*
  TI-84 Plus Silver Edition*
  TI-84 Plus C Silver Edition
  TI-84 Plus CE

** with the latest operating system (2.55MP) featuring MathPrint™ functionality.*



Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials
- Any required calculator files can be distributed to students via handheld-to-handheld transfer.

Lesson Files:

- Summing_Up_Geometric_Series_Student.pdf
- Summing_Up_Geometric_Series_Student.doc

Problem 1 – Infinite Series

An infinite series can be defined as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$, where a_1, a_2, and a_3 are terms of the series. Students begin by recognizing pattern within these series and writing *n*th term expression for a given series.

1. Find the next three terms of each infinite series.

a. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$

Answer:

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$

b. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$

Answer:

$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}$

c. $2+\frac{3}{2}+\frac{9}{8}+\frac{27}{32}\cdots$

Answer:

$2+\frac{3}{2}+\frac{9}{8}+\frac{27}{32}+\frac{81}{128}+\frac{243}{512}$

Students should see for question **1.c.** that the first term, 2, is the same as *a* in a geometric series, meaning $a + ar + ar^2 + \ldots + ar^n + \ldots$

2. Write an expression in terms of *n* that describes each of the above series using sigma notation.

Answers:

a. $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ or $\sum_{n=1}^{\infty}\frac{1}{2^n}$

b. $\sum_{n=1}^{\infty}\frac{n}{n+1}$

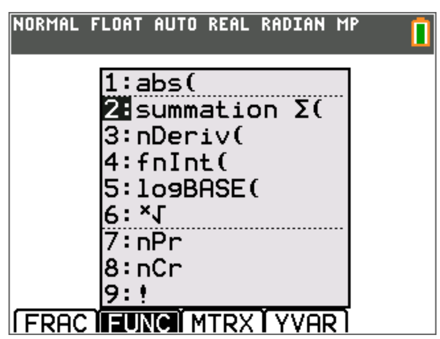
c. $\sum_{n=1}^{\infty}2\left(\frac{3}{4}\right)^{n-1}$

Problem 2 – Finding the Sum of a Geometric Series

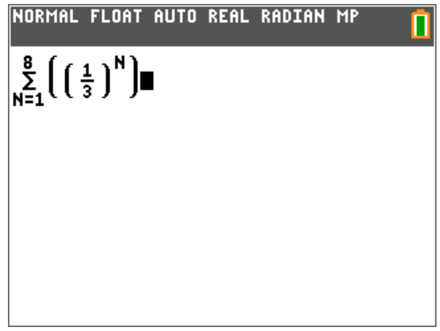
Students can find the partial sum of a geometric series. In this problem, students will find a partial sum of two geometric series.

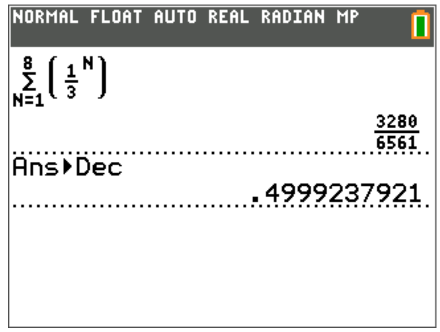
Press [alpha] [f2] [2] to select **2:summation Σ(**

Use the arrow keys to maneuver.



Find the partial sum of these geometric series. To find the sum of a series, press [alpha] [f2] [2] for summation. Use the arrow keys to maneuver. Notice that you need to type another set of parentheses within the parentheses that are supplied. To show the decimal, press [math] [2] [enter].





3a. $\sum_{n=1}^{8}\left(\frac{1}{3}\right)^{n} =$

Answer: $\frac{3280}{6561} \approx 0.499924$

3b. $\sum_{n=1}^{6}\left(\frac{1}{2}\right)^{n} =$

Answer: $\frac{63}{64} = 0.984375$

4. $\sum_{n=1}^{6} 2\left(\frac{3}{4}\right)^{n-1} =$

Answer: $\frac{3367}{512} \approx 6.57617$

Problem 3 – Convergence and Divergence of Geometric Series

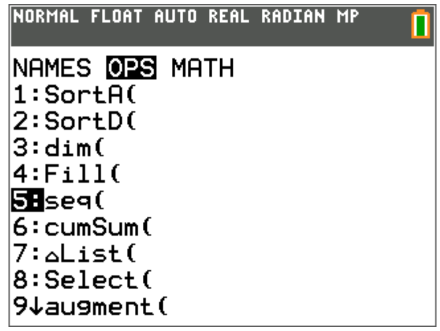
A geometric series with first term *a* and common ratio *r* is given by
$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ . \ . \ . \ + ar^n + \ . \ . \ . \ , a \neq 0$.

A geometric series diverges if $|r| \geq 1$. It converges to the sum $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ if $0 < |r| < 1$.

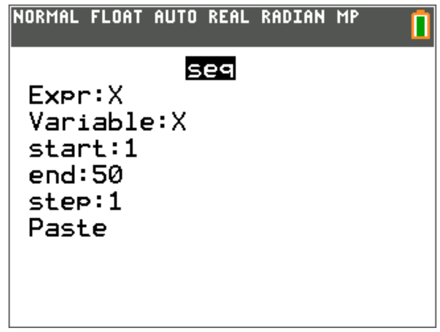
These conditions must be used in determining whether a series diverges or converges. It is also worth noting that a series may diverge, but will not necessarily diverge to infinity. A value *r* that is less than –1 will result in a series that diverges and has terms whose signs alternate from positive to negative, not diverging to infinity.

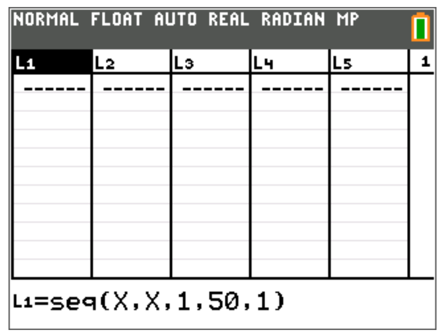
Another important note to students is that a series converges or diverges if the sequence of the partial sums converges to its sum or diverges.

Students will enter the values of a sequence in a list to further investigate the behavior of the geometric series. Press [stat] [enter] to access the table of data screen.

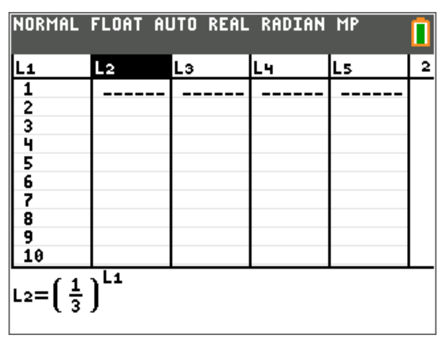


In **L1**, enter **seq(X,X,1,50)** in the top most cell. The **seq(** command can be found by pressing [2nd] [list] and arrowing over to **OPS** and selecting **5:seq(**. Enter the information in the **seq** menu exactly as shown in the screen to the right.



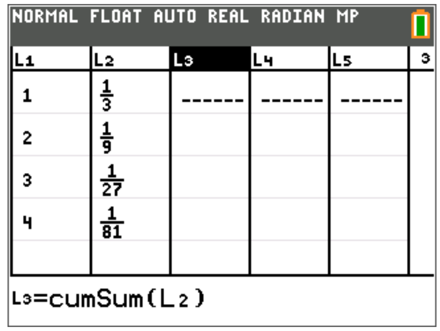


In the top most cell of **L2**, type $\left(\frac{1}{3}\right)^{L_1}$ and press [enter].

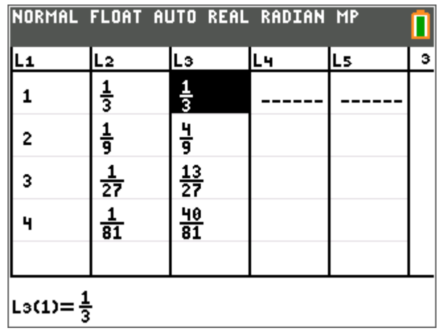


Students will now graph the series by generating a list with the cumulative sums of the terms of the sequence.

To do this, move to the top most cell of **L3**, press [enter], then press [2nd] [stat] [list] and arrow over to **OPS** and select **6:cumSum(**. Then type [2nd] [2] [L2] and press [enter].



The first 50 partial sums of the series $\sum_{n=1}^{50}\left(\frac{1}{3}\right)^n$ will be displayed in **L3**.

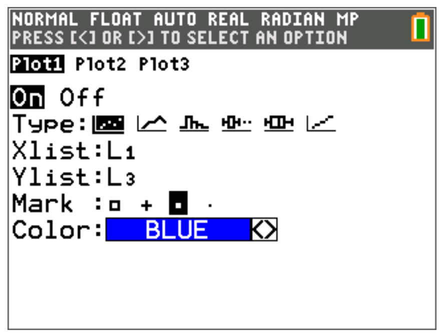


Next, we can view a graph for each series by creating a scatter plot of the values of the partial sums of the series.

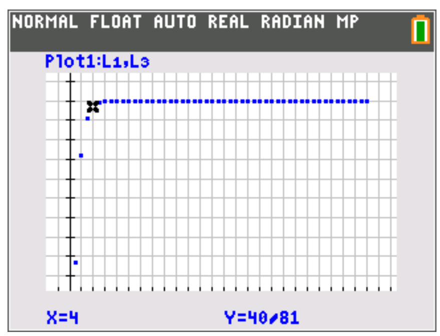
To create a scatter plot, select [2nd] [y=] [stat plot] [1].

Set up as shown in the figure to the right.

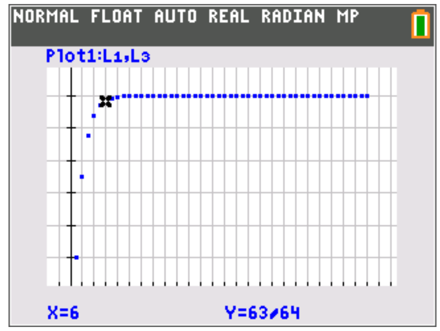
To view the graph, press [zoom] **9:ZoomStat**.

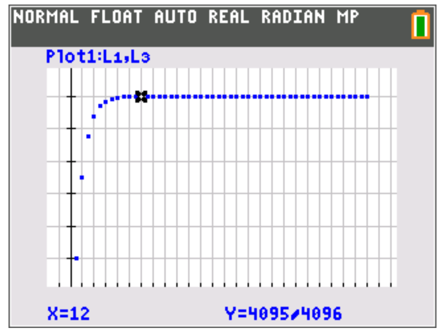


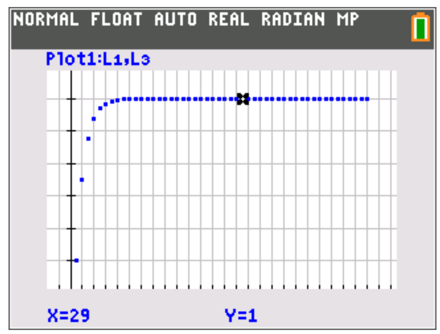
Pressing [window] and changing each of the following: **Xscl:** 2
Yscl: 0.2
It will give students a better view of the graph of the partial sums if grid lines are turned on.



Tech Tip: If your students are using the TI-84 Plus CE have them turn on the GridLine by pressing [2nd] [zoom] [format] to change the graph settings. If your students are using TI-84 Plus, they could use GridDot.







To find the sum of a geometric series, students must take the limit of the *n*th sum. The series $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ is a special case of the geometric series. One way of attempting this problem is to list the partial sums of the series then determine the *n*th sum.

The partial sums for this series would be
$S_1 = \frac{1}{2},\ S_2 = \frac{3}{4},\ S_3 = \frac{7}{8},\ S_4 = \frac{15}{16}, \ldots$

Guide students so they see that the denominator is 2 raised to the *n*th power and the numerator is always 1 less than the denominator. This gives $\frac{2^n - 1}{2^n}$.

Take the limit of the sum $\lim_{n\to\infty}\frac{2^n - 1}{2^n} = 1$.

By the geometric series test, the series converges because $0 < \frac{1}{2} < 1$, and the sum is 1, by finding the limit.

Another way is to use the geometric series test that can be stated $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ or $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. It is important to note to students the index of the series.

The three graphs to the right represent the series $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$. Notice that, when traced, the values of the series approach 1.

Determine the convergence or divergence of each of the following series. Create a scatter plot of the values or the partial sums to aid in determining the behavior of each series.

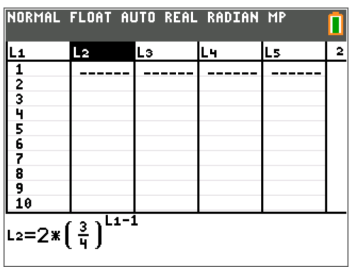
5. $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$

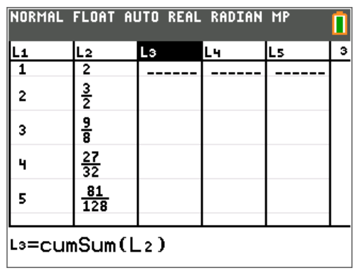
Answer: This series converges to 1.

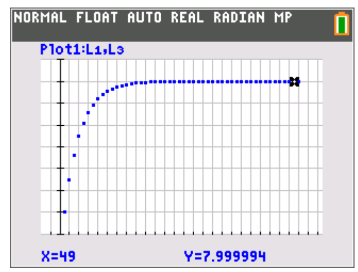
Repeating the same steps for 6 and 7 will yield the following results.

6. $\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^{n-1}$

Answer: This series converges to 8.

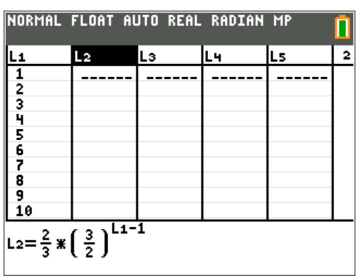


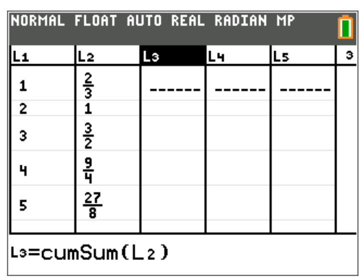


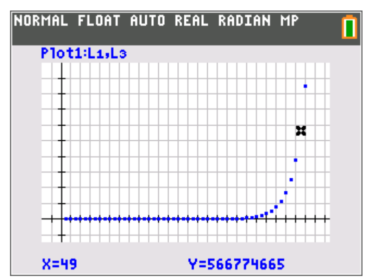


7. $\sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{3}{2}\right)^{n-1}$

Answer: The series diverges.







Have students sketch the graphs in the space provided on their worksheet.

Optional: Encourage students to participate in a class discussion.