Math Objectives

- Students will create a piecewise linear function to model a method for shuffling a deck of cards.
- Students will apply composite functions to represent two or more shuffles of a deck.
- Students will model with mathematics (CCSS Mathematical Practice).
- Students will use appropriate tools strategically (CCSS Mathematical Practice).

Vocabulary

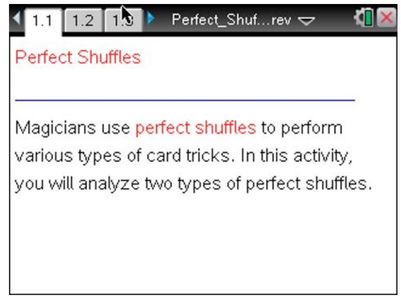
- perfect out-shuffle
- perfect in-shuffle
- piecewise linear function
- composition of functions

About the Lesson

- This lesson involves creating and using a composite function to describe certain ways of shuffling a deck of cards.
- As a result, students will:
  - Create a piecewise linear function to represent shuffling a deck of cards in a certain way.
  - Compose a piecewise function with itself several times.
  - Use a piecewise function and its composites to determine the number of shuffles needed to return a deck of cards to its original order.

TI-Nspire™ Navigator™ System

- Transfer a File.
- Use Screen Capture to examine patterns that emerge.
- Use Quick Poll to assess students' understanding.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Files:

Student Activity
Perfect_Shuffles_Student.pdf
Perfect_Shuffles_Student.doc

TI-Nspire document
Perfect_Shuffles.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

In a **perfect shuffle** of a deck of cards with an even number of cards, the magician splits the deck into an upper half and a lower half and then interlaces the cards alternately, one at a time from each half of the deck. For example, for a deck with 4 cards:

1 2 3 4

the split gives half decks 1 2 and 3 4. A perfect shuffle in which a card is taken first from the top half and then from the bottom half yields a new deck with cards in the order:

1 3 2 4

Repeating this process, the deck is split into 1 3 and 2 4 and the resulting shuffle yields a deck with cards in the order:

1 2 3 4

Two shuffles return the cards to their original order. This sequence in which a card is taken first from the **top** half is called an **out-shuffle** since the top and bottom cards remain on the outside of the deck.

Consider a deck with eight cards:

1 2 3 4 5 6 7 8.

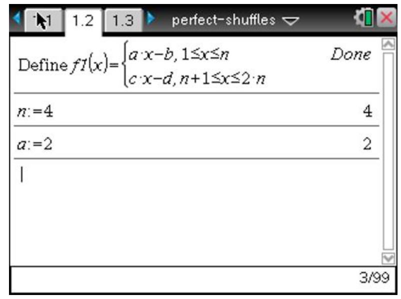
1. Complete the table below with the order of the eight cards from top to bottom after one and two perfect out-shuffles.

Answer:

original order [top-to-bottom]	1	2	3	4	5	6	7	8
order after one shuffle	**1**	**5**	**2**	**6**	**3**	**7**	**4**	**8**
order after two shuffles	**1**	**3**	**5**	**7**	**2**	**4**	**6**	**8**

Teacher Tip: Some students might enjoy using actual playing cards, or those made from 3”x 5” index cards, to learn about this out-shuffle.

Move to page 1.2.



This out-shuffle can be defined by a piece-wise linear function of the form $f1(x) = \begin{cases} a*x-b & 1 \le x \le 4 \\ c*x-d & 5 \le x \le 8 \end{cases}$ where each "piece" of the function corresponds to half of the deck. If x represents the position of a card before the shuffle, then $f1(x)$ represents its position after the shuffle.

2. The values *n* = 4 and a = 2 have been entered under the definition of $f1(x)$. Find and enter the values of *b*, *c*, and *d* for this out-shuffle.

Hint: To find *a* and *b*, use the facts that 1 → 1 and 2 → 3 so that *a*·1 – *b* = 1 and *a*·2 – *b* = 3.

<u>Answer:</u> $f1(x) = \begin{cases} 2*x-1 & 1 \le x \le 4 \\ 2*x-8 & 5 \le x \le 8 \end{cases}$.

Since *f*1(1) = 1 and *f*1(2) = 3, we have *a* – *b* = 1 and 2*a* – *b* = 3 so that *a* = 2 and *b* = 1. Similarly, *f1*(7) = 6 and *f*1(8) = 8, or 7*c* – *d* = 6 and 8*c* – *d* = 8 so *c* = 2 , *d* = 8.

Tech Tip: Be sure students can find := on the keyboard.

Move to page 1.3.

The position of each card after two shuffles is given by the composite function $f2(x) = f1(f1(x))$; if *x* represents the position of a card before the first shuffle, then $f2(x)$ represents its position after the second shuffle. The position of each card after three shuffles is found using $f3(x) = f1(f1(f1(x))) = f1(f2(x))$, and so on. Twelve shuffles have been defined on this page.

3. Express $f4(x)$ as the composition of two or more functions in two different ways.

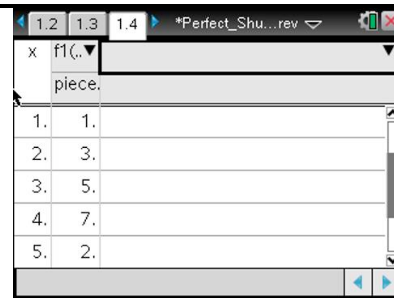
<u>Answer:</u>

$$f4(x) = f1(f1(f1(f1(x)))) = f1(f3(x)) = f2(f2(x))$$

Teacher Tip: Take special care to be sure students understand how the composite function f2(x) gives the positions of the cards after two shuffles, etc.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 1 at the end of this lesson.

Move to page 1.4.



The first two columns of the table should be identical to the first two rows of the table you completed in Question 1.

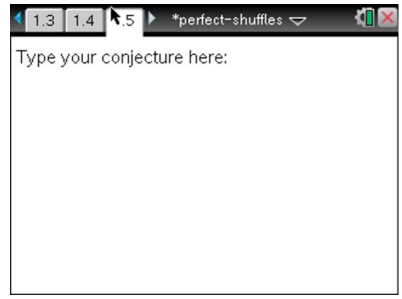
4. a. To perform the next shuffle, click on the black arrow at the top of the second column, and choose $f2$ from the menu of functions displayed. Verify that the entries in the second column are identical to those in the third row of your table from Question 1.

b. Move to the top of the third column. Click on the black arrow at the top of this column, and choose $f3$ from the menu of functions displayed. Examine these entries to check whether the cards have returned to their original order. If not, repeat this process, and choose $f4, f5, ...$ etc. until the entries in the column with the values of $fk(x)$ are the same as those in the first column giving the cards in their original order. (You might need to use the right arrow at the bottom of the page to display the next column.) How many out-shuffles were needed; i.e. what is the value of *k*?

Answer: Three out-shuffles were needed to return the deck to its original order; *k* = 3.

Teacher Tip: Tell students that when they reach a cell in a column of the spreadsheet whose position is greater than the number of cards in the deck being considered, they will see “UNDEF” or some similar symbol in those cells.

Move to page 1.5.



5. Make a conjecture about the smallest number of out-shuffles needed to return a deck of 2*n* cards to its original order when
   a. n = 8 [16 cards]:

   b. n = 16 [32 cards]:

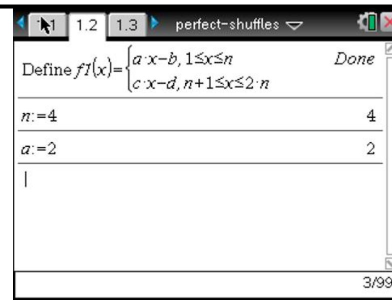
and type your conjecture in the box on Page 1.5.

Hint: In a deck with 2 cards, an out-shuffle does not change the order of the cards so 1 out-shuffle returns the cards to their original order. We have seen that for decks of 4 and 8 cards, 2 and 3 out-shuffles, respectively, are needed to return them to their original order.

Answer: 4 out-shuffles for a deck of 16 cards; 5 out-shuffles for a deck of 32 cards In general, *m* out-shuffles are needed to return a deck with 2^m cards to its original order.

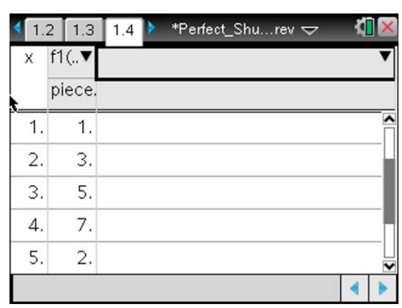
> **TI-Nspire Navigator Opportunity:** ***Screen Capture***
> **See Note 2 at the end of this lesson.**

Move back to page 1.2.



c. Suppose your conjecture for the number of out-shuffles needed to return the cards in a deck with 16 cards [n = 8] to their original positions was *k*. To check your conjecture. first, redefine $f1(x)$ for a deck with16 cards - enter *n* = 8 and any new values for *a*, *b*, *c*, and/or *d*.

Move to page 1.4.



Then return to the second column, and choose $f1$. In subsequent columns, successively choose $f2, f3, ...fk$ until finding the value of k showing that the cards have returned to their original order (for the first time). Was your conjecture correct?

Answer: $f1(x) = \begin{cases} 2*x-1 & 1 \le x \le 8 \\ 2*x-16 & 9 \le x \le 16 \end{cases}$.

Only the values of *n* and *d* need to be changed.

Answers vary on whether the conjecture was correct.

Teacher Tip: Students can delete any variable they are going to change and then enter the new value or enter the new values below the old ones.

Return to thinking about a four-card deck 1, 2, 3, 4. If the first card is taken from the **bottom** half, the following sequence is obtained:

1 2 3 4
3 1 4 2
4 3 2 1
2 4 1 3
1 2 3 4

Four shuffles are required to return the deck to its original order. This shuffle is an **in-shuffle**.

Again, consider a deck with eight cards:

1, 2, 3, 4, 5, 6, 7, 8

6. Follow the 'movement' of each of the two cards 3 and 4 through the deck after each in-shuffle. Stop when **both** cards have returned to their original positions. Complete the table showing the position of each card after 1, 2, 3, … shuffles and describe your reasoning.

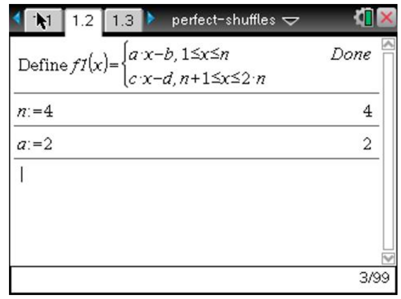
Answer:

number of shuffle]		1	2	3	4	5	6
position of 3	3	6	3	6	3	6	3
position of 4	4	8	7	5	1	2	4

Notice that the positions of card 3 repeat every two shuffles while those of card 4 do so every 6 shuffles. A student could shuffle an actual deck of cards, imagine shuffling a deck of cards, or do the solution to the next problem (b) and record the answers obtained

Teacher Tip: Some students might enjoy using actual playing cards, or those made from 3"x 5" index cards, to learn about this in-shuffle.

Move back to page 1.2.

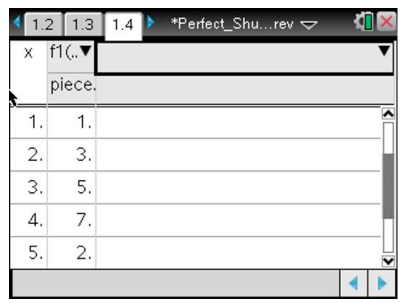


7. Let *k* be the number of in-shuffles needed to return the cards in this deck to their original positions. To determine the value of *k*:

Teacher Tip: Students can delete any variable they are going to change and then enter the new value or enter the new values below the old ones.

Redefine f $f1(x)$ for an in-shuffle on a deck with 8 cards - enter *n* = 4 and then any new values for *a*, *b*, *c*, and/or *d*.

Move to page 1.4.



Then return to the second column, and choose f1. In subsequent columns, successively choose $f2, f3, ..., fk$ until you find the value of k showing that the cards have returned to their original order (for the first time). What is the value of *k*?

<u>**Answer:**</u> $f1(x) = \begin{cases} 2*x & 1 \le x \le 4 \\ 2*x-9 & 5 \le x \le 8 \end{cases}$.

so that k = 6.

Teacher Tip: Insert teacher/Tech tip as needed.

Move to page 1.5.

8. Make a conjecture about the smallest number of in-shuffles needed to return a deck of 2*n* cards to its original order when
   a. n = 8 [16 cards]
   b. n = 16 [32 cards].

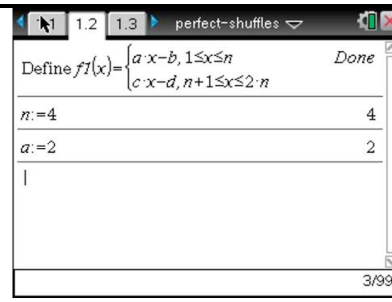
And type your conjecture in the box on Page 1.5.

Hint: For decks of cards with 2, 4, 8, 16, .. cards, how does the number of out-shuffles compare to the number of in-shuffles?

Answer: 8 in-shuffles for a deck of 16 cards; 10 in-shuffles for a deck of 32 cards In general, 2*m* in-shuffles are needed to return a deck with 2^m cards to its original order.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 3 at the end of this lesson.

Move back to page 1.2.



9. It can be shown that it takes 52 in-shuffles to return the cards in a standard deck of 52 cards to their original order. Determine your answer by using the same process of redefining $f1$ for an out-shuffle of a deck with 52 cards on page 1.2 and then finding the number out-shuffles needed on page 1.4. How many out-shuffles are needed?

   Answer: 8 out-shuffles for to return a standard deck of 52 cards to its original order.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students are able to understand:

- That repeated shuffles of certain types of a deck of cards can be represented by composing a piecewise linear function with itself several times.

Assessment

- Ask students to find minimum number of out-shuffles and in-shuffles needed to return a deck of 10 cards and/or 14 cards to its original order. [10 cards: (6,10); 14 cards (12,4)]
- Ask the students why 52 [*m*] is the maximum number of in-shuffles or out-shuffles needed to return deck of 52 [*m*] cards to its original order.

TI-Nspire Navigator

Note 1 Question 2, Name of Feature: Quick Poll Use a Quick Poll to determine students' understanding of the definition of the composite function f2(x)

Note 2 Question 5, Name of Feature: Screen Capture Use a Screen Capture to share students' conjectures on the minimum number of out-shuffles needed to return a deck of 16 cards and 32 cards to their original orders.

Note 3 Question 8, Name of Feature: Screen Capture Use a Screen Capture to share students' conjectures on the minimum number of in-shuffles needed to return a deck of 16 cards and 32 cards to their original orders.