Math Objectives

- Given a graph of a function, students will be able to determine over what intervals the definite integral of that function will be positive, negative, or zero.
- Students will be able to make generalizations about the behavior of the definite integral of any continuous function.
- Students will be able to explain the mathematical deficiency in the "area under the curve" description of the definite integral.
- Students will construct viable arguments and critique the reasoning of others. (CCSS Mathematical Practice)
- Students will reason abstractly and quantitatively. (CCSS Mathematical Practice)

Vocabulary

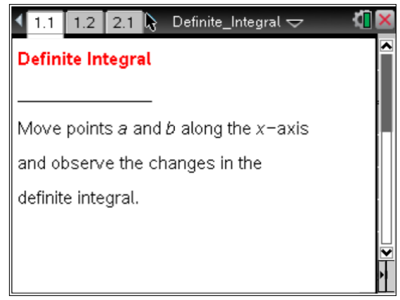
- signed area
- definite integral
- continuous function

About the Lesson

- The intent of this lesson is to help students make visual connections between the definite integral of a function and the signed area between the function and the *x*-axis. In particular, this lesson provides opportunities to develop a mathematically accurate conception of the definite integral that avoids the "area under the curve" description of the definite integral.
- Using an odd function and an even function, students change the bounds of the definite integral of the function and observe a visual representation of the accumulated signed area and the value of the definite integral.
- The lesson provides opportunities to make generalizations about the definite integrals of the given functions and about continuous functions.

TI-Nspire™ Navigator™ System

- Use Screen Capture and Quick Poll to inform the teacher of the students' understanding as they respond to the questions posed on the student activity worksheet.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- In the *Graphs & Geometry* application, you can hide the function entry line by pressing ctrl G.

Lesson Materials:

Student Activity

- Definite_Integral_Student.pdf
- Definite_Integral_Student.doc

TI-Nspire document

- Definite_Integral.tns

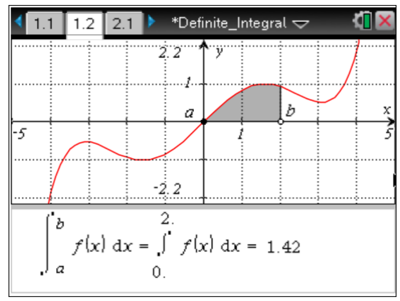
Visit www.mathnspired.com for lesson updates and tech tip videos.

Discussion Points and Possible Answers

> **Tech Tip:** If students experience difficulty dragging a point, check to make sure that they have moved the cursor until it becomes a hand (👆) getting ready to grab the point. Also, be sure that the word *point* appears, not the word *text*. Then press ctrl 👆 to grab the point and close the hand (✊).

Move to page 1.2.

1. The graph shown is of the function $y = f(x)$. The definite integral of $f(x)$ from a to b is given by $\int_a^b f(x)\,dx$. For example, $\int_0^2 f(x)\,dx$ is the definite integral of $f(x)$ from 0 to 2, or between $x = 0$ and $x = 2$.

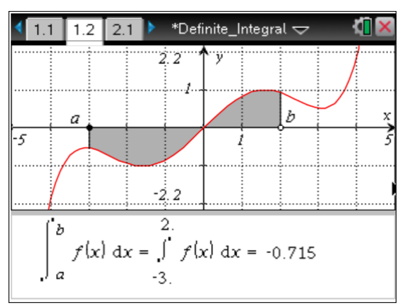


Drag points a and b along the x-axis to determine the values of the following definite integrals, where f is the function shown in the graph.

a. $\int_0^2 f(x)\,dx =$ ________________

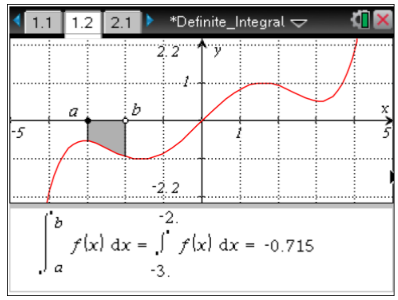
<u>Answer:</u> 1.42

b. $\int_{-3}^2 f(x)\,dx =$ ________________



<u>Answer:</u> –0.715

c. $\int_{-3}^{-2} f(x)\,dx =$ _______________



Answer: –0.715

Teacher Tip: You may need to make sure that students are setting *a* and *b* properly. Students may make the mistake of reversing the limits, resulting in incorrect answers.

TI-Nspire Navigator Opportunity: *Screen Capture*
See Note 1 at the end of the lesson.

2. Drag point *a* to –3 and move point *b* to determine the following:
   a. For what values of *b* is $\int_{-3}^{b} f(x)\,dx$ positive? What do you observe about the shaded region and the graph of *f* when $\int_{-3}^{b} f(x)\,dx$ is positive?

Answer: $\int_{-3}^{b} f(x)\,dx$ is positive for |b| > 3. For b > 3, the area between *f*(*x*) and the *x*-axis lying above the *x*-axis is greater than the area between *f*(*x*) and the *x*-axis lying below the *x*-axis. For b < –3, the area is still below the *x*-axis, but the area is accumulating in the negative direction.

   b. For what values of *b* is $\int_{-3}^{b} f(x)\,dx$ negative? What do you observe about the shaded region and the graph of *f* when $\int_{-3}^{b} f(x)\,dx$ is negative?

Answer: $\int_{-3}^{b} f(x)\,dx$ is negative for –3 < *b* < 3. For values of *b* in this range, the area between *f*(*x*) and the *x*-axis lying below the *x*-axis is greater than the area between *f*(*x*) and the *x*-axis lying above the *x*-axis.

   c. For what values of *b* does $\int_{-3}^{b} f(x)\,dx = 0$? What do you observe about the shaded region and the graph of *f* when $\int_{-3}^{b} f(x)\,dx = 0$?

Answer: $\int_{-3}^{b} f(x)\,dx = 0$ when *b* = –3 or *b* = 3. When *b* = –3, there is no area accumulated because the interval over which we are integrating has no length. When *b* = 3, the area between *f*(*x*) and the *x*-axis lying above the *x*-axis is equal to the area between *f*(*x*) and the *x*-axis lying below the *x*-axis.

> **Teacher Tip:** You may need to encourage students to explore values of *b* that are less than, greater than, or equal to –3. The case of the negative-valued definite integral resulting from an upper limit of the integral that is less than the lower limit of the integral is a unique case that may merit further discussion with your students. You may want to note that, much as we consider the area between the function and the *x*-axis to be signed, so do we consider the lengths of intervals over which we integrate to be signed.

3. For the function *f* pictured on page 1.2, under what conditions of *a* and *b* in [–5, 5] will the definite integral $\int_a^b f(x)\,dx$ be positive? Negative? Zero? Explain your thinking.

Answer: The integral will be positive either when *a* is negative and *b* < *a* (the case of the negative signed interval over which we are integrating *and* area lying below the *x*-axis) or when |*b*| > |*a*|. The second case results in a positive integral in one of three ways: because all the accumulated area is above the *x*-axis (both *a* and *b* are positive); because the area above the *x*-axis is greater than that below (*a* is negative and *b* > –*a*); because the area above the *x*-axis is less than that below, but the area takes the opposite sign because *b* is less than *a*.

The integral will be negative either when *a* is positive and –*a* < *b* < *a* (this is the case of integration over a negative signed interval when area above the *x*-axis is greater than the area below the *x*-axis) or when *a* is negative and *a* < *b* < –*a.* The second case allows for more accumulated area under the *x*-axis than above.

The integral will be 0 when either *a* = *b* or *b* = –*a*. When *a* = *b*, no area accumulates under the curve because there is no interval over which to integrate. When *b* = –*a*, an equal amount of area lies above the *x*-axis as below. This second case is unique for odd functions.

> **Teacher Tip:** Be sure that students explore values of *a* and *b* with *a* < –*b*, *a* > –*b*, and *a* = –*b*. Understanding the change in sign of the integral caused by integrating over an interval of negative length can be subtle. You may need to guide your students through this idea.

Move to page 2.2.

4. The graph on page 2.2 is of a new function *f* and the definite integral $\int_a^b f(x)\,dx$. Drag point *a* to –3 (if *a* is not already positioned at –3).

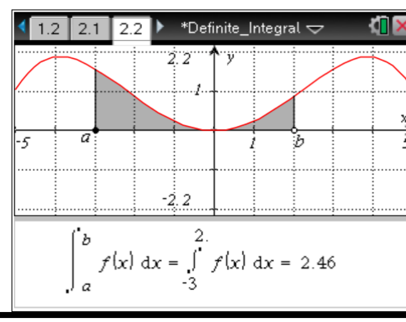
   a. Without dragging point *b*, for what values of *b* do you think $\int_{-3}^b f(x)\,dx$ will be positive? Negative? Zero? Explain your predictions.

   <u>Answer:</u> The integral should be positive for *b* > 3, negative when *b* < –3, and 0 when *b* = –3. Because the whole function is above the *x*-axis, the area will be above the *x*-axis and so the integral will always be positive if the interval is oriented in the positive direction (with *b* > –3). It will be negative only if the interval is directed negatively (with *b* < –3), and it will be 0 only if the interval length is 0, or when *b* = –3.

   b. Drag point *b* to test your predictions. Describe what you observed in the graph of *f* that confirmed or contradicted your prediction.

   <u>Answer:</u> Student predictions will vary.

Teacher Tip: This page gives students an opportunity to explore an even function after having observed an odd function. Be sure students predict before testing predictions using the document.



TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 2 at the end of the lesson.

5. For the function *f*(*x*) pictured on page 2.2, under what conditions of *a* and *b* in [–5, 5] will the definite integral $\int_a^b f(x)\,dx$ be positive? Negative? Zero? Explain your thinking.

   <u>Answer:</u> The integral will be positive for *a* < *b*, because all the area is above the *x*-axis and the interval over which the function is integrated has positive length. The integral will be negative when *b* < *a* because all the area is above the *x*-axis and the interval is of negative length. The integral will be 0 only when *b* = *a* because with all the area above the *x*-axis, there is no way for area to cancel out.

6. Based on your observations on pages 1.2 and 2.2, for any continuous function *f* on an interval [*c*, *d*] and for *a* and *b* in [*c*, *d*] when will the definite integral be positive? Negative? Zero? Clearly explain your generalization.

   Answer: In general, a definite integral of a continuous function will be positive for $a < b$ when the area between the function and the *x*-axis lying below the *x*-axis is less than the area between the function and the *x*-axis that lies above the *x*-axis. If $a > b$, the area above the *x*-axis must be less than that below the *x*-axis for the integral to be positive.

   The integral of a continuous function will be negative for $a < b$ when the area between the function and the *x*-axis lying below the *x*-axis is greater than the area between the function and the *x*-axis that lies above the *x*-axis. If $a > b$, the area above the *x*-axis must be greater than that below the *x*-axis for the integral to be positive.

   The integral of a continuous function will be 0 if $a = b$ or if the area between the function and the *x*-axis lying below the *x*-axis is equal to the area between the function and the *x*-axis that lies above the *x*-axis.

7. The definite integral is often described as "the area under the curve $y = f(x)$ between $x = a$ and $x = b$." What problems do you see with this definition?

   Answer: Student answers may vary. The intention here is for students to understand that the idea of accumulated, signed area is important to understanding the definite integral. We generally do not consider area to be negative, so the "area under the curve" description does not provide consideration for the difference between area above the *x*-axis and area below. Furthermore, area under the curve can be very confusing when the curve lies below the *x*-axis, creating the misconception that one is always "measuring down" from the function.

Wrap Up

Upon completion of this activity, the teacher should ensure that students understand:

- How to determine the sign of the definite integral based on the graph of a function.
- Under what conditions a definite integral of a continuous function will be positive, negative, or 0.
- The deficiencies in the "area under the curve" description of the definite integral.

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Note 1

Question 1, *Screen Capture*:

After students have had time to respond to items 1a–1c, pose the following task: Locate a and b such that the value of the definite integral is 2.14.

Display the screen captures and discuss the different intervals that result in a value of 2.14. Check specifically whether any of the responses have an example showing $b < a$. If this does not occur naturally with this investigation, you may wish to point it out now or wait until it presents itself again later in this activity.

Note 2

Question 4, *Quick Poll*:

After students have had time to respond to items 4a and 4b, pose the following task: If $a = 1$ on the graph shown on page 2.2 of the .tns file, what values of b would result in a value of the definite integral that is less than 0? If this poll indicates lack of understanding, refocus students on integrating over an interval of negative length.