Take it to the limit?

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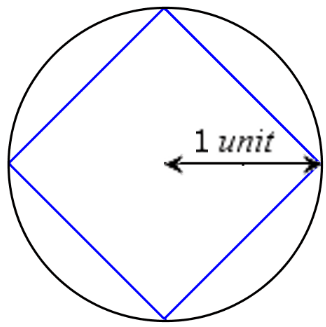
Student Activity

7 8 9 10 **11** 12

TI-Nspire | Investigation | Student | 45 min

Introduction

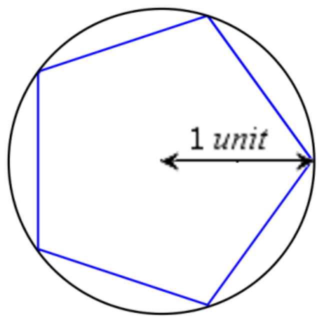
Suppose you didn't know the formula for the circumference of a circle, you could approximate the circumference by calculating the perimeter of a square that just fits inside the circle. A better approximation could be achieved by using a regular pentagon, and better again using a regular hexagon. This investigation explores the perimeter of a regular *n* sided polygon that just fits inside a circle with radius 1 unit.



Question: 1

Determine the perimeter of the square shown and state whether this approximation is too big or too small in relation to circumference of the circle.

Answer: $4\sqrt{2}$ Note that students could calculate this answer using Pythagoras' theorem or trigonometry.



Question: 2

By dividing a regular pentagon into 10 right angled triangles, determine the perimeter of the pentagon and compare the result with the circumference of the circle that 'circumscribes' the pentagon.

Answer: By constructing 10 congruent right angled triangles, students can use trigonometry to show that the base of each is $\sin\left(\frac{2\pi}{10}\right) \approx 0.58779$. The total perimeter of the polygon is therefore: $P \approx 5.8779$.

Question: 3

By dividing a regular hexagon into 12 right angled triangles, determine the perimeter of the hexagon and compare the result the circumference of the circle that 'circumscribes' the hexagon.

Answer: By constructing 12 congruent right angled triangles, students can use trigonometry to show that the base of each is $12\sin\left(\frac{2\pi}{12}\right) = 6$. The circumference of the circle is 2π.

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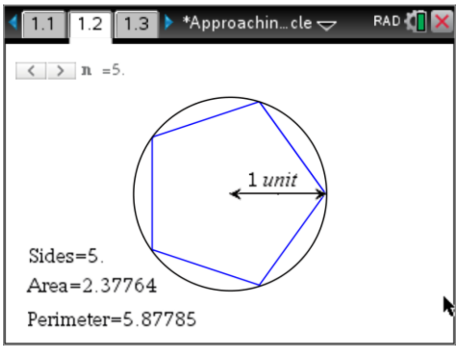

Load the TI-Nspire file “Take it to the limit” and navigate to page 1.2. Use the slider to change the value of *n* and check your answers for the square, pentagon and hexagon.

Continue increasing the value of *n* and identify the limiting value for the perimeter of an *n* sided regular polygon.

Check the graph on page 1.3 that shows a record of the number of sides and the corresponding circumference.

Question: 4

What is the limiting value for the perimeter of an *n* sided regular polygon? Will this limit ever be reached or exceeded? Explain.

Comment: The purpose of this question is to drive student understanding with regards to the concept of a limit in preparation for the gradient at a point. By providing a practical, geometric approach, students can visualise the sides of the polygon getting smaller and smaller as *n* increases, but they are still ‘straight lines’. It is clear that the perimeter will get close to 2π and will never exceed... it is the ‘limiting’ value. This understanding can then be drawn upon when students are required to work out the gradient at a point, when the *run* approaches zero.

Answer: As the number of sides increases ($n \to \infty$) the perimeter approaches 2π. As $n \neq \infty$ the perimeter of the polygon cannot equal 2π however the limiting value is 2π.

Question: 5

Write a rule for the perimeter of an *n* sided regular polygon and define it as a function $f(n)$ in the calculator application on page 2.1. Use the ‘limit’ command in the calculus menu to determine the limit of your rule as $n \to \infty$. Note that the document is set to ‘radians’ so the rule, wherever trigonometric functions are involved, should take this into account.

Syntax: $\lim\limits_{n\to\infty} f(n)$

Answer: $f(n) = 2n\sin\left(\frac{\pi}{n}\right)$ and $\lim\limits_{n\to\infty} f(n) = 2\pi$ If students use degrees: $f(n) = 2n\sin\left(\frac{180^\circ}{n}\right)$, the degree sign can also be used on the calculator to override the radians setting, however it is possible to use the Taylor polynomial for sine to show students why the limit approaches 2π.

As a Taylor polynomial $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\ldots$ so for $x = \frac{\pi}{n}$ the rule becomes:

$$2n\sin\left(\frac{\pi}{n}\right) = 2n\left(\frac{\pi}{n} - \frac{\pi^3}{3!n^3} + \frac{\pi^5}{5!n^5} - \frac{\pi^7}{7!n^7}\ldots\right)$$

$$= 2\pi - \frac{2\pi^3}{3!n^2} + \frac{2\pi^5}{5!n^4} - \frac{2\pi^7}{7!n^6}\ldots$$

The first term is independent of *n* all subsequent terms involve a division by *n* so as *n* increases the subsequent terms approach zero.

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In this investigation *n* is used in the slider in problem 1 for the number of sides for the regular polygon. This means it is assigned a value, making it a parameter rather than a true variable. By using a **New Problem** within the document all links to *n* are cleared for problem 2 returning *n* to variable status. Adding problems to a document allows multiple uses of the same variable. The page number is now 2.1 signifying it is problem 2, page 1.

Question: 6

Another way to approximate the circumference of the circle is to put a regular polygon around the outside of the circle.

a. Determine the perimeter of a square that just fits around the outside of the circle.

Side lengths = 2, so perimeter = 8.

b. Determine the perimeter of regular pentagon that just fits around the outside of the circle.

Perimeter of regular pentagon: ≈ 7.26543. An exact calculation is possible but not necessary.

c. Define a rule for an *n* sided regular polygon and determine the limit of this function as $n \to \infty$

$$f(n) = 2n\tan\left(\frac{\pi}{n}\right)$$

d. Will the perimeter of the outer polygon ever equal the circumference of the circle?

As before, as $n \to \infty$ the perimeter approaches 2π, but never equals the circumference of the circle.

Page 3.1 contains a dynamic representation of the inner and outer polygons, use this representation to check your equation for the perimeter of the polygon that just fits around the circle. The graph on page 3.2 shows how both functions approach the same limit, but from different directions.

Question: 7

The polygons can be used to approximate the area of the circle. Page 3.1 includes a dynamic representation of this including the corresponding values for the area. The area produced by the inner (inscribed) polygon produces a smaller estimate for the area. The outer (circumscribed) polygon produces a larger estimate. Determine the respective rules for the area of the inscribed and circumscribed polygons and show that they approach the same limit as $n \to \infty$.

Inscribed Polygon Area: $\frac{n}{2}\sin\left(\frac{2\pi}{n}\right)$ (Note that there are variations of this result)

Circumscribed Polygon Area: $n\tan\left(\frac{\pi}{n}\right)$

The level of proof is up to the teacher's requirements as the Taylor polynomial of sine can easily demonstrate the limit. The limit command from the calculus menu recognises the limit for both functions.

Comment: This question also acts as a reference point to show that the limit approached from 'both sides' is the same… to help support the definition of differentiability.

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