Spirally Surds

Teacher Notes & Answers

7 **8 9 10** 11 12

TI-Nspire CAS   Investigation   Teacher   60+ min

Aim

To use manual and TI-Nspire CAS technology to explore the visual and numerical patterns in the Spiral of Theodorus, and the role that surds play in measurement predictions.

Equipment

For this activity you will need:

- TI-Nspire CAS

- TI-Nspire CAS document SpirallySurds.tns

- Ruler, unruled A4 or A3 paper, protractor, colouring pens/pencils
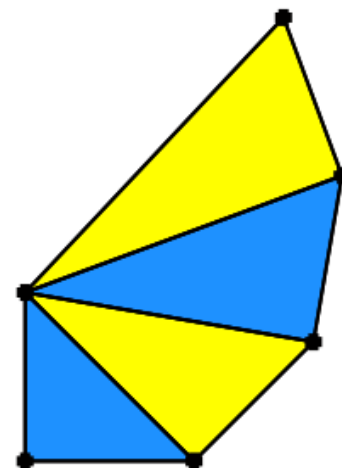
Introduction

The Spiral of Theodorus (aka Einstein's Spiral, the Pythagorean Spiral) is a beautiful example of an aesthetically-pleasing "curve" can be formed with adjacent right-angled triangles. In this activity, teachers can choose to mix the hands-on artwork with the TI-Nspire CAS mathematical explorations in proportions dictated by student interest and understanding.

The worldwide web has many links to hand drawing the Spiral of Theodorus, one such being https://www.youtube.com/watch?v=Yr4if2mUTjk&ab_channel=PedroGalindo. While the teacher in this video uses imperial measurements, the practicalities of using a fixed length larger than the centimetre are worth considering. The resulting artwork this teacher's students produce may provide some ideas about how your students can display their learning in a creative, physical form.

We're going to look at how the Geometry application of TI-Nspire can be used to accurately reproduce the Spiral of Theodorus in a manner that is artistically pleasing to many students. At the same time, the precision afforded by the technology will enable students to predict the limits to the ever-growing spiral posed by the paper or screen they are using to create it. We'll use the Lists & Spreadsheet application to store successive length and area calculations to assist that process.

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TEXAS INSTRUMENTS

General information about the spiral

- We start with a vertical line segment AB, with A above B, of any convenient length which we will call 1 unit.

- We then form an isosceles right-angled triangle ABC where $\overline{AC}$ is the hypotenuse, and $\overline{BC}$ is also of length 1 unit.

- We then construct another line segment of length 1 unit called $\overline{CD}$; this is perpendicular to $\overline{AC}$.

- Again, we complete the triangle by constructing its hypotenuse, $\overline{AD}$.

- This process continues...

- Measurements of hypotenuse, perimeter and area of each triangle may use either hand-measured distances (converted to "units") or, for those who know Pythagoras' Rule and surd arithmetic, irrational exact values may be found; patterns may also be apparent to some.
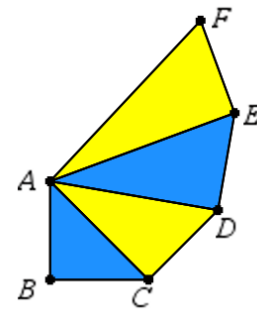
Advice about by-hand construction

Teachers may choose to have some or all students try their hands at manual drawing first; a 2cm or 3cm length for "1 unit" may work better on an A3 sheet of paper. Accuracy in estimating the right angle, as well as precision in measuring a consistent "unit", will vary significantly as the spiral grows; this may provide a nice contrast to the technology-assisted construction (shown below) and lead to good discussion regarding accuracy and aesthetics. Colour schemes used could reflect student preference (footy team colours, etc).

Questions arising from by-hand construction:

Question 1.

How many triangles will it take to make one full rotation (or, before overlapping occurs)?

16

Question 2.

After how many triangles will the spiral spill off the paper?

Answers will vary – depending on accuracy of hand-measured components. If "1 unit" = 2cm, and point A is at the centre of a sheet of paper measuring 20.7cm by 29.3cm, then there will be 37 complete triangles before the edge of the paper is reached.

Question 3.

What percentage of the paper's surface area will be covered by the spiral before overlapping occurs?

Answers will vary. Using the materials with scale and dimensions described above, the spiral will cover between 14% and 15% of the paper.

Question 4.

If the first triangle is Triangle 1, the second is Triangle 2, etc – can you find a pair of triangles where the larger triangle has exactly double the area of the smaller triangle? Exactly triple the area? Are these pairs unique?

Triangles 4 and 1, 8 and 2, 12 and 3, 16 and 4 etc have a 2:1 area ratio; Triangles 9 and 3, 18 and 2, 27 and 9 etc have a 3:1 area ratio. Therefore, there is a predictable pattern locating these non-unique pairs.

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Part I - Using TI-Nspire to construct the spiral

Open the TI-Nspire file SpirallySurds.tns.

Note that p1.1 is a Geometry application with triangle ABC constructed and coloured blue. There is a slider on the screen which controls the size of what we've called "1 unit" that ranges in value from 1 to 5, in increments of 0.1; use the cursor to see the effect of changing the slider's value.
Set the cursor to a value of 4 before starting the next step; this will allow students to see what they're doing more clearly.

Using **Menu-Shapes-Circle**, select point C, then select the slider (this will create a circle centred at C with a radius of 4).

Be sure to press Esc to leave this construction tool.

Next, use **Menu-Construction-Perpendicular** and select $\overline{AC}$ and then select point C; this will create a line which forms a right angle to $\overline{AC}$ at C.

Press Esc.

Next, use **Menu-Points & Lines – Intersection Point(s)** and click on first the new line and then the circle. We'll be using the point above and to the right of C, and hiding the other intersection point and the constructions used to produce it.

Since we have the key point we'll label "D", we now wish to "hide" the other unnecessary items on the screen – the circle, the perpendicular line, and the <u>other</u> intersection point. We can do this by selecting each one of these (say, the circle) and then using **Ctrl-Menu-Hide/Show – Hide Selection**, and it will disappear; do the same with the line and extraneous point.

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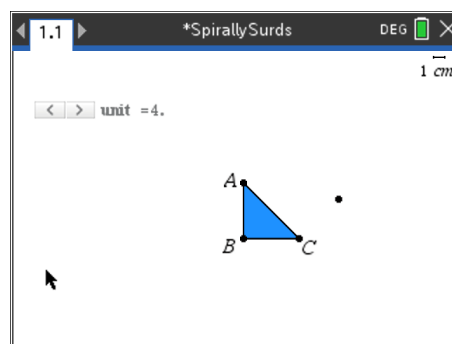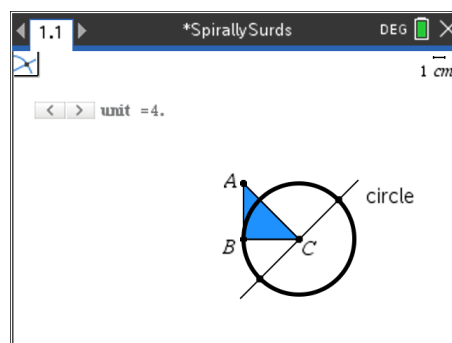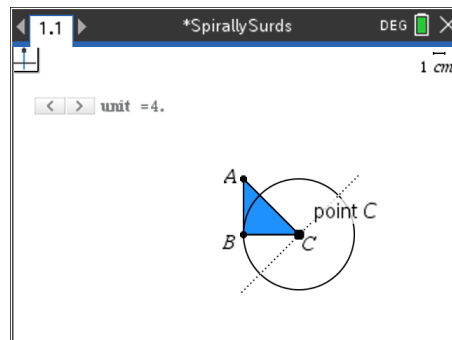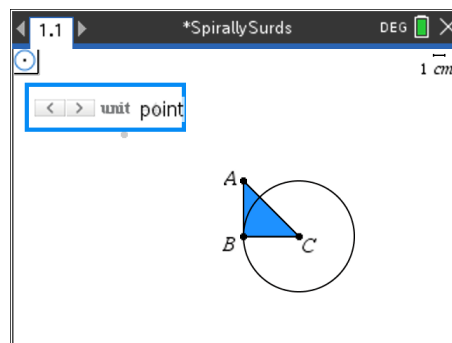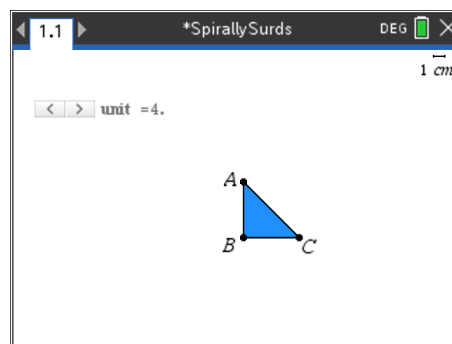
To label this point as D, select the point and use **Ctrl-Menu-Label** and type D in the text box, press Enter.
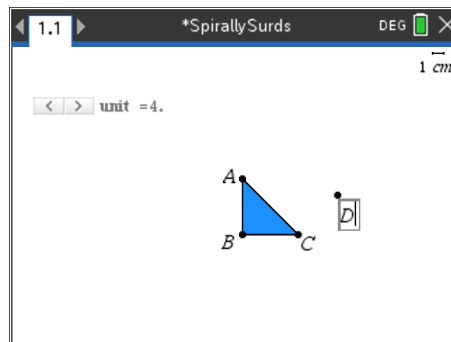
To construct side $\overline{CD}$, use **Menu-Points & Lines – Segment** and click on each of points C and D. Repeat this to construct the hypotenuse $\overline{AD}$. The triangle ACD will need to be constructed as well (for measuring area and perimeter, as well as colouring in). Use **Menu-Shapes-Triangle** and select the three points in any order.
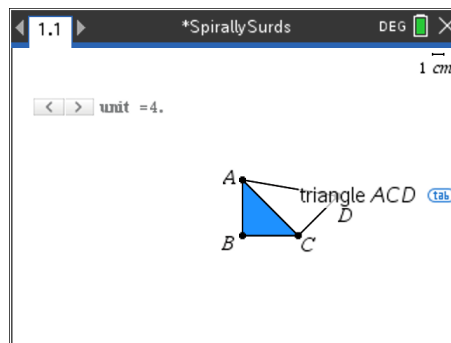
To colour the interior of the triangle, move the cursor onto the shape then press **Ctrl-Menu-Color-Fill Color** and select the one you want. Now adjust the slider to change the apparent length of "1 unit". Repeat the process for successive triangles.
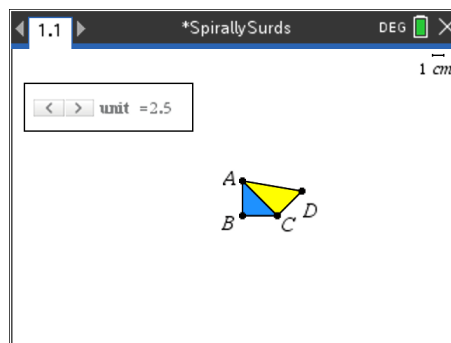
Questions arising from construction using TI-Nspire:

Question 5.

With the slider set to 2, how many triangles will fit on the handheld screen?

30 complete triangles

Question 6.

What slider setting will allow for 1 full rotation of the spiral? (Note: You can adjust the slider's settings by moving the cursor over the slider and pressing Ctrl-Menu-Settings etc)

Between 2.5 and 2.6

Now try animating the slider using the same adjustment path described above.

Question 7.

By using Menu-Measurement, try measuring the area, length and angle within your construction. Note that for overlapping constructions, you'll need tab to toggle between say triangle ACD and side $\overline{AD}$ depending on what the cursor is resting on or pointing to. Does this vary with the slider setting?

Yes, the slider setting affects the measurements shown; note the "1cm" scale in top right corner.

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TEXAS INSTRUMENTS

Part II – Using TI-Nspire to analyse the measurements

We'll now look at p1.2 of our file which is a Lists & Spreadsheets application. Some of the initial entries have been made already, they'll be explained below. Teaching points include:

- For any triangle in the spiral, the lengths of the perpendicular sides (in units) will be 1 and the length of previous (old) hypotenuse.
- The hypotenuse of the current (new) triangle is the square root of ($1^2 + (\text{old hypotenuse})^2$).
- The area of the triangle is half of the product of the perpendicular sides – which are always 1 and the previous (old) hypotenuse.

On p1.2 note that Column A is labelled with the list variable *Triangle* and the first four entries (for the first four triangles of the spiral) are shown. As they form an arithmetic sequence, this list can be extended. Column B is labelled *hyp_tri* and will be the hypotenuse of that particular triangle. Thus, in row 3 this will be the hypotenuse length for the third triangle, etc. Column C, labelled *area_tri*, will contain the area of each triangle. The fourth column D will calculate the ratio of the area of any triangle to that of the previous (adjacent) triangle (see Q8, p5). Row 1 entries for Triangle 1 have been entered manually in columns B and C.

Here (and throughout the rest of p1.2) the entries are calculated by formula. Entries for row 2 of the file are entered by formula. The formula used for cell B2 (displayed at the bottom of the screen) shows the hypotenuse for triangle 2 is calculated using Pythagoras' Rule, with the constant side (1 unit) and the cell B1 value ($\sqrt{2}$ unit), the hypotenuse of the previous triangle. Formulae for cells C2 and D2 are $= \dfrac{1 \times b1}{2}$ and $= \dfrac{c2}{c1}$, respectively. Note: you need the "=" sign to start any formula, and for column C division by 2 is needed (as opposed to multiplication by 0.5, which would result in a decimal approximation).

Now with formulae in cells B2, C2 and D2, these may be copied for subsequent rows, allowing for the analysis of patterns suggested by previous questions. Place cursor in lower right corner of the cell you wish to copy, noting the white cross that appears, and drag down to row 4 as shown.

It is suggested that the list in column A be extended first, say to 20, or 50. After that, the formula copying in columns B, C and D could be done.

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TEXAS INSTRUMENTS

Questions arising from formula analysis using TI-Nspire:

Question 8.

Consider the hypotenuse values in column B. Can you predict the length of the hypotenuse for triangle 100? How might this knowledge help you determine the answer to Q2 on p2?

Yes, the hypotenuse for Triangle 100 is $\sqrt{101}$; in general, the nth triangle has a hypotenuse of length $\sqrt{n+1}$ units. You could, after centring point A on your paper, measure the shortest distance to the sides and top/bottom edges of the paper; then compare these distances (converted to "units") to the ever-increasing value in Column B. Care must be taken, however, to ascertain the angle at which these hypotenuse segments are emanating from point A, as larger values than those distances are possible.

Question 9.

Some triangles have rational numbers for their hypotenuse lengths, while many have irrational (surd) numbers. Why does this occur?

From the generalisation above, for n-values which are 1 less than a perfect square (such as 8, or 63) the value of $\sqrt{n+1}$ is rational.

Question 10.

Using the exact values seen in column D (once the formulae have been copied for say, the first 50 triangles), can you find any of the pairs of triangles with double (or triple) areas, as seen in Q8 p5?

Yes. For example (using the notation ATn for the area of triangle n), AT8=$\sqrt{2}$ which is twice the value of AT2 ($\frac{\sqrt{2}}{2}$). Exact surd values from the Lists & Spreadsheet application, rather than approximate decimal values as shown on the Geometry application, make the comparisons of these areas much easier.

Question 11.

For students familiar with the inverse trigonometric functions: what formula might be used in column E from cell E2 downwards that calculates the size of the angle at A for each triangle of the Spiral of Theodorus? You may then be able to answer Q1 on page 2 of this sheet.

For any triangle, the angle at A is such that $\sin A = \frac{1}{hypotenuse}$ and therefore $A = \sin^{-1}(\frac{1}{hypotenuse})$.

Under a heading such as **angle_at_a** for column E, the formula $= \sin^{-1}(\frac{1}{[b1]})$ placed in cell E1 and then copied down to cell E16 (see answer for Q2) will produce a list variable **angle_at_a** which has 16 values. Open a Calculator application, press **var** and select **angle_at_a** and press **Enter** and the values will appear in a comma-separated set. Then type **sum(angle_at_a)** and press **Enter** again, and the values for the angle at A in the first rotation of the spiral will be added; press **Ctrl-Enter** and the decimal approximation will show 351.15, just short of 360°.

Teachers wishing to further extend the investigation to include bivariate statistics might add a Data & Statistics application as p1.3 to compare the list variables (for example, *area_tri* and *triangle*) graphically.

Teachers wishing to contact me about this activity and how they have used it with their classes are encouraged to do so at rogerdwander@gmail.com